

# TRACE IDEALS FOR FOURIER INTEGRAL OPERATORS WITH NON-SMOOTH SYMBOLS II

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**ABSTRACT.** We consider Fourier integral operators with symbols in modulation spaces and non-smooth phase functions whose second orders of derivatives belong to certain types of modulation space. We establish continuity and Schatten-von Neumann properties of such operators when acting on modulation spaces.

## 0. INTRODUCTION

In [6], A. Boulkhemair considers a certain class of Fourier integral operators where the corresponding symbols are defined without any explicit regularity assumptions and with only small regularity assumptions on the phase functions. The symbol class here is, in the present paper, denoted by  $M^{\infty,1}$  and contains  $S_{0,0}^0$ , the set of smooth functions which are bounded together with all their derivatives. In the time-frequency community,  $M^{\infty,1}$  is known as a modulation space. (See e. g. [12, 14, 19] or the definition below.) Boulkhemair then proves that such operators are uniquely extendable to continuous operators on  $L^2$ . In particular it follows that pseudo-differential operators with symbols in  $M^{\infty,1}$  are  $L^2$ -continuous, which was proved by J. Sjöstrand in [28], where it seems that  $M^{\infty,1}$  was used for the first time in this context.

More recent contribution to the theory of Fourier integral operators with non-smooth symbols are presented in [24–26]. For example, in [25], Ruzhansky and Sugimoto investigate, among others,  $L^2$  estimates for Fourier integral operators with symbols in local Sobolev-Kato spaces, and with less regularity assumptions on the phase functions comparing to [6].

In this paper we consider Fourier integral operators where the symbol classes are given by  $M^{p,q}$  where  $p, q \in [1, \infty]$ , and with phase functions satisfying similar conditions as in [6]. We discuss continuity of such operators when acting on modulation spaces, and prove Schatten-von Neumann properties when acting on  $L^2$ .

In order to be more specific we recall some definitions. Assume that  $p, q \in [1, \infty]$  and that  $\omega \in \mathcal{P}(\mathbf{R}^{2n})$  (see Section 1 for the definition of  $\mathcal{P}(\mathbf{R}^n)$ ). Then the *modulation space*  $M_{(\omega)}^{p,q}(\mathbf{R}^n)$  is the set of all  $f \in \mathcal{S}'(\mathbf{R}^n)$  such that

$$(0.1) \quad \|f\|_{M_{(\omega)}^{p,q}} \equiv \left( \int \left( \int |\mathcal{F}(f\tau_x\chi)(\xi)\omega(x,\xi)|^p dx \right)^{q/p} d\xi \right)^{1/q} < \infty$$

(with obvious modification when  $p = \infty$  or  $q = \infty$ ). Here  $\tau_x$  is the translation operator  $\tau_x\chi(y) = \chi(y - x)$ ,  $\mathcal{F}$  is the Fourier transform on  $\mathcal{S}'(\mathbf{R}^n)$  which is given by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-n/2} \int f(x)e^{-i\langle x,\xi \rangle} dx$$

when  $f \in \mathcal{S}(\mathbf{R}^n)$ , and  $\chi \in \mathcal{S}(\mathbf{R}^n) \setminus \{0\}$  is called a *window function* which is kept fixed. For convenience we set  $M_{(\omega)}^{p,q} = M_{(\omega)}^{p,q}$  when  $\omega = 1$ .

During the last twenty years, modulation spaces have been an active fields of research (see e. g. [12–15, 19, 22, 32, 35]). They are rather similar to Besov spaces

(see [2, 30, 35] for sharp embeddings) and it has appeared that they are useful to have in background in time-frequency analysis and to some extent also in pseudo-differential calculus.

Next we discuss the definition of Fourier integral operators. For conveniency we restrict ourselves to operators which belong to  $\mathcal{L}(\mathcal{S}(\mathbf{R}^n), \mathcal{S}'(\mathbf{R}^n))$ . Here we let  $\mathcal{L}(V_1, V_2)$  denote the set of all linear and continuous operators from  $V_1$  to  $V_2$ , when  $V_1$  and  $V_2$  are topological vector spaces. For any appropriate  $a \in \mathcal{S}'(\mathbf{R}^{2n+m})$  (the *symbol*) and real-valued  $\varphi \in C(\mathbf{R}^{2n+m})$  (the *phase function*), the Fourier integral operator  $\text{Op}_\varphi(a)$  is defined by the formula

$$(0.2) \quad \text{Op}_\varphi(a)f(x) = (2\pi)^{-n} \iint a(x, y, \xi) f(y) e^{i\varphi(x, y, \xi)} dy d\xi,$$

when  $f \in \mathcal{S}(\mathbf{R}^n)$ . Here the integrals should be interpreted in distribution sense if necessary. By letting  $m = n$ , and choosing symbols and phase functions in appropriate ways, it follows that the pseudo-differential operator

$$\text{Op}(a)f(x) = (2\pi)^{-n} \iint a(x, y, \xi) f(y) e^{i\langle x-y, \xi \rangle} dy d\xi$$

is a special case of Fourier integral operators. Furthermore, if  $t \in \mathbf{R}$  is fixed, and  $a$  is an appropriate function or distribution on  $\mathbf{R}^{2n}$  instead of  $\mathbf{R}^{3n}$ , then the definition of the latter pseudo-differential operators cover the definition of pseudo-differential operators of the form

$$(0.3) \quad a_t(x, D)f(x) = (2\pi)^{-n} \iint a((1-t)x + ty, \xi) f(y) e^{i\langle x-y, \xi \rangle} dy d\xi.$$

On the other hand, in the framework of harmonic analysis it follows that the map  $a \mapsto a_t(x, D)$  from  $\mathcal{S}(\mathbf{R}^{2n})$  to  $\mathcal{L}(\mathcal{S}(\mathbf{R}^n), \mathcal{S}'(\mathbf{R}^n))$  is uniquely extendable to a bijection from  $\mathcal{S}'(\mathbf{R}^{2n})$  to  $\mathcal{L}(\mathcal{S}(\mathbf{R}^n), \mathcal{S}'(\mathbf{R}^n))$ . Consequently, any Fourier integral operator is equal to a pseudo-differential operator of the form  $a_t(x, D)$ .

In the literature it is usually assumed that  $a$  and  $\varphi$  in (0.2) are smooth functions. For example, if  $a \in \mathcal{S}'(\mathbf{R}^{2n+m})$  and  $\varphi \in C^\infty(\mathbf{R}^{2n+m})$  satisfy  $\partial^\alpha \varphi \in S_{0,0}^0(\mathbf{R}^{2n+m})$  when  $|\alpha| = N$  for some integer  $N \geq 0$ , then it is easily seen that  $\text{Op}_\varphi(a)$  is continuous on  $\mathcal{S}(\mathbf{R}^n)$  and extends to a continuous map from  $\mathcal{S}'(\mathbf{R}^n)$  to  $\mathcal{S}'(\mathbf{R}^n)$ . In [1] it is proved that if  $\partial^\alpha \varphi \in S_{0,0}^0(\mathbf{R}^{2n+m})$  when  $|\alpha| = 2$  and satisfies

$$(0.4) \quad \left| \det \begin{pmatrix} \varphi''_{x,y} & \varphi''_{x,\xi} \\ \varphi''_{y,\xi} & \varphi''_{\xi,\xi} \end{pmatrix} \right| \geq d$$

for some  $d > 0$ , then the definition of  $\text{Op}_\varphi$  extends uniquely to any  $a \in S_{0,0}^0(\mathbf{R}^{2n+m})$ , and then  $\text{Op}_\varphi(a)$  is continuous on  $L^2(\mathbf{R}^n)$ . Next assume that  $\varphi$  instead satisfies  $\partial^\alpha \varphi \in M^{\infty,1}(\mathbf{R}^{3n})$  when  $|\alpha| = 2$  and that (0.4) holds for some  $d_0$ . This implies that the condition on  $\varphi$  is relaxed since  $S_{0,0}^0 \subseteq M^{\infty,1}$ . Then Boulkhemair improves the result in [1] by proving that the definition of  $\text{Op}_\varphi$  extends uniquely to any  $a \in M^{\infty,1}(\mathbf{R}^{2n+m})$ , and that  $\text{Op}_\varphi(a)$  is still continuous on  $L^2(\mathbf{R}^n)$ .

In Section 2 we discuss Schatten-von Neumann properties for Fourier integral operators which are related to those which were considered by Boulkhemair. More precisely, let  $p' \in [1, \infty]$  denote the conjugate exponent of  $p \in [1, \infty]$ . Then we prove that if  $\omega_1, \omega_2 \in \mathcal{P}(\mathbf{R}^{2n})$  and  $\omega \in \mathcal{P}(\mathbf{R}^{4n})$  are appropriate weight functions,  $p, q \in [1, \infty]$  are such that  $q \leq \min(p, p')$  and  $a \in M_{(\omega)}^{p,q}(\mathbf{R}^{2n})$  then  $\text{Op}_\varphi(a)$  belongs to  $\mathcal{I}_p(\mathcal{H}_1, \mathcal{H}_2)$ , the set of Schatten-von Neumann operators of order  $p \in [1, \infty]$  from the Hilbert space  $\mathcal{H}_1 = M_{(\omega_1)}^2$  to  $\mathcal{H}_2 = M_{(\omega_2)}^2$ . Recall that an operator  $T$  from  $\mathcal{H}_1$  to  $\mathcal{H}_2$  is a Schatten-von Neumann operator of order  $p$  if it is linear and

continuous from  $\mathcal{H}_1$  to  $\mathcal{H}_2$ , and satisfies

$$\|T\|_{\mathcal{S}_p} \equiv \sup \left( \sum |(Tf_j, g_j)_{\mathcal{H}_2}|^p \right)^{1/p} < \infty.$$

Here the supremum should be taken over all orthonormal sequences  $(f_j)$  in  $\mathcal{H}_1$  and  $(g_j)$  in  $\mathcal{H}_2$ .

Furthermore, assume that  $p, q \in [1, \infty]$  are such that  $\frac{1}{p} + \frac{1}{q} = 1$  and  $p \leq p', m = n$  and instead  $a(x, y, \xi) = b(x, \xi)$ , for some  $b \in M^{p, q}(\mathbf{R}^{2n})$ , and that in addition

$$(0.5) \quad |\det(\varphi''_{y, \xi})| \geq d$$

holds for some constant  $d > 0$ . Then we prove that  $\text{Op}_\varphi(a) \in \mathcal{S}_p$ . When proving these results we first prove that they hold in the case  $p = 1$ . The remaining cases are then consequences of Boukhemair's result, interpolation and duality.

## 1. PRELIMINARIES

In this section we discuss basic properties for modulation spaces. The proofs are in many cases omitted since they can be found in [10–16, 19, 33–35].

We start by discussing some notations. The duality between a topological vector space and its dual is denoted by  $\langle \cdot, \cdot \rangle$ . For admissible  $a$  and  $b$  in  $\mathcal{S}'(\mathbf{R}^n)$ , we set  $(a, b) = \langle a, \bar{b} \rangle$ , and it is obvious that  $(\cdot, \cdot)$  on  $L^2$  is the usual scalar product.

Next assume that  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are topological spaces. Then  $\mathcal{B}_1 \hookrightarrow \mathcal{B}_2$  means that  $\mathcal{B}_1$  is continuously embedded in  $\mathcal{B}_2$ . In the case that  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are Banach spaces,  $\mathcal{B}_1 \hookrightarrow \mathcal{B}_2$  is equivalent to  $\mathcal{B}_1 \subseteq \mathcal{B}_2$  and  $\|x\|_{\mathcal{B}_2} \leq C\|x\|_{\mathcal{B}_1}$ , for some constant  $C > 0$  which is independent of  $x \in \mathcal{B}_1$ .

Next we discuss appropriate conditions for the involved weight functions. Let  $\omega, v \in L^\infty_{loc}(\mathbf{R}^n)$  be positive functions. Then  $\omega$  is called *v-moderate* if

$$(1.1) \quad \omega(x + y) \leq C\omega(x)v(y), \quad x, y \in \mathbf{R}^n,$$

for some constant  $C > 0$ , and if  $v$  in (1.1) can be chosen as a polynomial, then  $\omega$  is called *polynomially moderated*. Furthermore,  $v$  is called *submultiplicative* if (1.1) holds for  $\omega = v$ . We denote by  $\mathcal{P}(\mathbf{R}^n)$  the set of all polynomially moderated functions on  $\mathbf{R}^n$ .

Assume that  $p, q \in [1, \infty]$ , and that  $\chi \in \mathcal{S}(\mathbf{R}^n) \setminus 0$ . Then recall that the *modulation space*  $M^{p, q}_{(\omega)}(\mathbf{R}^n)$  is the set of all  $f \in \mathcal{S}'(\mathbf{R}^n)$  such that (0.1) holds. We note that the definition of  $M^{p, q}_{(\omega)}(\mathbf{R}^n)$  is independent of the choice of window  $\chi$ , and that different choices of  $\chi$  give rise to equivalent norms. (See Proposition 1.1 below.) For convenience we set  $M^p_{(\omega)} = M^{p, p}_{(\omega)}$ . Furthermore, if  $\omega \equiv 1$  we also set  $M^{p, q} = M^{p, q}_{(\omega)}$ .

The following proposition is a consequence of well-known facts in [12, 19]. Here and in what follows, we let  $p'$  denote the conjugate exponent of  $p$ , i. e.  $1/p + 1/p' = 1$  should be fulfilled.

**Proposition 1.1.** *Assume that  $p, q, p_j, q_j \in [1, \infty]$  for  $j = 1, 2$ , and  $\omega, \omega_1, \omega_2, v \in \mathcal{P}(\mathbf{R}^{2n})$  are such that  $\omega$  is  $v$ -moderate and  $\omega_2 \leq C\omega_1$  for some constant  $C > 0$ . Then the following are true:*

- (1) *if  $\chi \in M^1_{(v)}(\mathbf{R}^n) \setminus 0$ , then  $f \in M^{p, q}_{(v)}(\mathbf{R}^n)$  if and only if (0.1) holds, i. e.  $M^{p, q}_{(\omega)}(\mathbf{R}^n)$  is independent of the choice of  $\chi$ . Moreover,  $M^{p, q}_{(\omega)}$  is a Banach space under the norm in (0.1), and different choices of  $\chi$  give rise to equivalent norms;*
- (2) *if  $p_1 \leq p_2$  and  $q_1 \leq q_2$  then*

$$\mathcal{S}(\mathbf{R}^n) \hookrightarrow M^{p_1, q_1}_{(\omega_1)}(\mathbf{R}^n) \hookrightarrow M^{p_2, q_2}_{(\omega_2)}(\mathbf{R}^n) \hookrightarrow \mathcal{S}'(\mathbf{R}^n);$$

- (3) the  $L^2$  product  $(\cdot, \cdot)$  on  $\mathcal{S}$  extends to a continuous map from  $M_{(\omega)}^{p,q}(\mathbf{R}^n) \times M_{(1/\omega)}^{p',q'}(\mathbf{R}^n)$  to  $\mathbf{C}$ . On the other hand, if  $\|a\| = \sup |(a, b)|$ , where the supremum is taken over all  $b \in \mathcal{S}(\mathbf{R}^n)$  such that  $\|b\|_{M_{(1/\omega)}^{p',q'}} \leq 1$ , then  $\|\cdot\|$  and  $\|\cdot\|_{M_{(\omega)}^{p,q}}$  are equivalent norms;
- (4) if  $p, q < \infty$ , then  $\mathcal{S}(\mathbf{R}^n)$  is dense in  $M_{(\omega)}^{p,q}(\mathbf{R}^n)$ . The dual space of  $M_{(\omega)}^{p,q}(\mathbf{R}^n)$  can be identified with  $M_{(1/\omega)}^{p',q'}(\mathbf{R}^n)$ , through the form  $(\cdot, \cdot)_{L^2}$ . Moreover,  $\mathcal{S}(\mathbf{R}^n)$  is weakly dense in  $M_{(\omega)}^\infty(\mathbf{R}^n)$ .

Proposition 1.1 (1) permits us be rather vague concerning the choice of  $\chi \in M_{(v)}^1 \setminus 0$  in (0.1). For example, if  $C > 0$  is a constant and  $\Omega$  is a subset of  $\mathcal{S}'$ , then  $\|a\|_{M_{(\omega)}^{p,q}} \leq C$  for every  $a \in \Omega$ , means that the inequality holds for some choice of  $\chi \in M_{(v)}^1 \setminus 0$  and every  $a \in \Omega$ . Evidently, for any other choice of  $\chi \in M_{(v)}^1 \setminus 0$ , a similar inequality is true although  $C$  may have to be replaced by a larger constant, if necessary.

It is also convenient to let  $\mathcal{M}_{(\omega)}^{p,q}(\mathbf{R}^n)$  be the completion of  $\mathcal{S}(\mathbf{R}^n)$  under the norm  $\|\cdot\|_{M_{(\omega)}^{p,q}}$ . Then  $\mathcal{M}_{(\omega)}^{p,q} \subseteq M_{(\omega)}^{p,q}$  with equality if and only if  $p < \infty$  and  $q < \infty$ . It follows that most of the properties which are valid for  $M_{(\omega)}^{p,q}(\mathbf{R}^n)$ , also hold for  $\mathcal{M}_{(\omega)}^{p,q}(\mathbf{R}^n)$ .

We also need to use multiplication properties of modulation spaces. The proof of the following proposition is omitted since the result can be found in [12, 14, 34, 35].

**Proposition 1.2.** Assume that  $p, p_j, q_j \in [1, \infty]$  and  $\omega_j, v \in \mathcal{P}(\mathbf{R}^{2n})$  for  $j = 0, \dots, N$  satisfy

$$\frac{1}{p_1} + \dots + \frac{1}{p_N} = \frac{1}{p_0}, \quad \frac{1}{q_1} + \dots + \frac{1}{q_N} = N - 1 + \frac{1}{q_0},$$

and

$$\omega_0(x, \xi_1 + \dots + \xi_N) \leq C \omega_1(x, \xi_1) \dots \omega_N(x, \xi_N), \quad x, \xi_1, \dots, \xi_N \in \mathbf{R}^n,$$

for some constant  $C$ . Then  $(f_1, \dots, f_N) \mapsto f_1 \dots f_N$  from  $\mathcal{S}(\mathbf{R}^n) \times \dots \times \mathcal{S}(\mathbf{R}^n)$  to  $\mathcal{S}(\mathbf{R}^n)$  extends uniquely to a continuous map from  $M_{(\omega_1)}^{p_1, q_1}(\mathbf{R}^n) \times \dots \times M_{(\omega_N)}^{p_N, q_N}(\mathbf{R}^n)$  to  $M_{(\omega_0)}^{p_0, q_0}(\mathbf{R}^n)$ , and

$$\|f_1 \dots f_N\|_{M_{(\omega_0)}^{p_0, q_0}} \leq C \|f_1\|_{M_{(\omega_1)}^{p_1, q_1}} \dots \|f_N\|_{M_{(\omega_N)}^{p_N, q_N}}$$

for some constant  $C$  which is independent of  $f_j \in M_{(\omega_j)}^{p_j, q_j}(\mathbf{R}^n)$  for  $i = 1, \dots, N$ .

Furthermore, if  $u_0 = 0$  when  $p < \infty$ ,  $v(x, \xi) = v(\xi) \in \mathcal{P}(\mathbf{R}^n)$  is submultiplicative,  $f \in M_{(v)}^{p, 1}(\mathbf{R}^n)$ , and  $\phi, \psi$  are entire functions on  $\mathbf{C}$  with expansions

$$\phi(z) = \sum_{k=0}^{\infty} u_k z^k, \quad \psi(z) = \sum_{k=0}^{\infty} |u_k| z^k,$$

then  $\phi(f) \in M_{(v)}^{p, 1}(\mathbf{R}^n)$ , and

$$\|\phi(f)\|_{M_{(v)}^{p, 1}} \leq C \psi(C \|f\|_{M_{(v)}^{p, 1}}),$$

for some constant  $C$  which is independent of  $f \in M_{(v)}^{p, 1}(\mathbf{R}^n)$

**Remark 1.3.** Assume that  $p, q, q_1, q_2 \in [1, \infty]$ ,  $\omega_1 \in \mathcal{P}(\mathbf{R}^n)$  and that  $\omega, v \in \mathcal{P}(\mathbf{R}^{2n})$  are such that  $\omega$  is  $v$ -moderate. Then the following properties for modulation spaces hold:

- (1) if  $q_1 \leq \min(p, p')$ ,  $q_2 \geq \max(p, p')$  and  $\omega(x, \xi) = \omega_1(x)$ , then  $M_{(\omega)}^{p, q_1} \subseteq L_{(\omega_0)}^p \subseteq M_{(\omega)}^{p, q_2}$ . In particular,  $M_{(\omega)}^2 = L_{(\omega_0)}^2$ ;
- (2) if  $\omega(x, \xi) = \omega_1(x)$ , then  $M_{(\omega)}^{p, q}(\mathbf{R}^n) \hookrightarrow C(\mathbf{R}^n)$  if and only if  $q = 1$ ;
- (3)  $M^{1, \infty}$  is a convolution algebra which contains all measures on  $\mathbf{R}^n$  with bounded mass;
- (4) if  $x_0 \in \mathbf{R}^n$  and  $\omega_0(\xi) = \omega(x_0, \xi)$ , then  $M_{(\omega)}^{p, q} \cap \mathcal{E}' = \mathcal{FL}_{(\omega_0)}^q \cap \mathcal{E}'$ . Furthermore, if  $B$  is a ball with radius  $r$  and center at  $x_0$ , then

$$C^{-1} \|\widehat{f}\|_{L_{(\omega_0)}^q} \leq \|f\|_{M_{(\omega)}^{p, q}} \leq C \|\widehat{f}\|_{L_{(\omega_0)}^q}, \quad f \in \mathcal{E}'(B)$$

for some constant  $C$  which only depends on  $r, n, \omega$  and the chosen window functions;

- (5) if  $\omega(x, \xi) = \omega(\xi, x)$ , then  $M_{(\omega)}^p$  is invariant under the Fourier transform. A similar fact holds for partial Fourier transforms;
- (6) for each  $x, \xi \in \mathbf{R}^n$  we have

$$\|e^{i\langle \cdot, \xi \rangle} f(\cdot - x)\|_{M_{(\omega)}^{p, q}} \leq C v(x, \xi) \|f\|_{M_{(\omega)}^{p, q}},$$

for some constant  $C$ ;

- (7) if  $\tilde{\omega}(x, \xi) = \omega(x, -\xi)$  then  $f \in M_{(\omega)}^{p, q}$  if and only if  $\bar{f} \in M_{(\tilde{\omega})}^{p, q}$ .

(See e. g. [10–12, 14–16, 19, 35].)

For future references we note that the constant  $C_{r, n}$  is independent of the center of the ball  $B$  in (4) in Remark 1.3.

In our investigations we need the following characterization of modulation spaces.

**Proposition 1.4.** *Let  $\{x_\alpha\}_{\alpha \in I}$  be a lattice in  $\mathbf{R}^n$ ,  $B_\alpha = x_\alpha + B$  where  $B \subseteq \mathbf{R}^n$  is an open ball, and assume that  $f_\alpha \in \mathcal{E}'(B_\alpha)$  for every  $\alpha \in I$ . Also assume that  $p, q \in [1, \infty]$ . Then the following is true:*

- (1) if

$$(1.2) \quad f = \sum_{\alpha \in I} f_\alpha \quad \text{and} \quad F(\xi) \equiv \left( \sum_{\alpha \in I} |\widehat{f}_\alpha(\xi) \omega(x_\alpha, \xi)|^p \right)^{1/p} \in L^q(\mathbf{R}^n),$$

then  $f \in M_{(\omega)}^{p, q}$ , and  $f \mapsto \|F\|_{L^q}$  defines a norm on  $M_{(\omega)}^{p, q}$  which is equivalent to  $\|\cdot\|_{M_{(\omega)}^{p, q}}$  in (0.1);

- (2) if in addition  $\cup_\alpha B_\alpha = \mathbf{R}^n$ ,  $\chi \in C_0^\infty(B)$  satisfies  $\sum_\alpha \chi(\cdot - x_\alpha) = 1$ ,  $f \in M_{(\omega)}^{p, q}(\mathbf{R}^n)$ , and  $f_\alpha = f \chi(\cdot - x_\alpha)$ , then  $f_\alpha \in \mathcal{E}'(B_\alpha)$  and (1.2) is fulfilled.

*Proof.* (1) Assume that  $\chi \in C_0^\infty(\mathbf{R}^n) \setminus 0$  is fixed. Since there is a bound of overlapping supports of  $f_\alpha$ , we obtain

$$\begin{aligned} |\mathcal{F}(f \chi(\cdot - x))(\xi) \omega(x, \xi)| &\leq \sum |\mathcal{F}(f_\alpha \chi(\cdot - x))(\xi) \omega(x, \xi)| \\ &\leq C \left( \sum |\mathcal{F}(f_\alpha \chi(\cdot - x))(\xi) \omega(x, \xi)|^p \right)^{1/p}. \end{aligned}$$

for some constant  $C$ . From the support properties of  $\chi$ , and the fact that  $\omega$  is  $v$ -moderate for some  $v \in \mathcal{P}(\mathbf{R}^{2n})$ , it follows for some constant  $C$  independent of  $\alpha$  we have

$$|\mathcal{F}(f_\alpha \chi(\cdot - x))(\xi) \omega(x, \xi)| \leq C |\mathcal{F}(f_\alpha \chi(\cdot - x))(\xi) \omega(x_\alpha, \xi)|.$$

Hence, for some balls  $B'$  and  $B'_\alpha = x_\alpha + B'$ , we get

$$\begin{aligned}
& \left( \int |\mathcal{F}(f\chi(\cdot - x))(\xi)\omega(x, \xi)|^p dx \right)^{1/p} \\
& \leq C \left( \sum_\alpha \int_{B'_\alpha} |\mathcal{F}(f_\alpha\chi(\cdot - x))(\xi)\omega(x_\alpha, \xi)|^p dx \right)^{1/p} \\
& \leq C \left( \sum_\alpha \int_{B'_\alpha} (|\widehat{f}_\alpha\omega(x_\alpha, \cdot)| * |\widehat{\chi}v(0, \cdot)|(\xi))^p dx \right)^{1/p} \\
& \leq C'' \left( \sum_\alpha (|\widehat{f}_\alpha\omega(x_\alpha, \cdot)| * |\widehat{\chi}v(0, \cdot)|(\xi))^p \right)^{1/p} \leq C'' F * |\widehat{\chi}v(0, \cdot)|(\xi),
\end{aligned}$$

for some constants  $C'$  and  $C''$ . Here we have used Minkowski's inequality in the last inequality. By applying the  $L^q$ -norm and using Young's inequality we get

$$\|f\|_{M_{(\omega)}^{p,q}} \leq C'' \|F * |\widehat{\chi}v(0, \cdot)|\|_{L^q} \leq C'' \|F\|_{L^q} \|\widehat{\chi}v(0, \cdot)\|_{L^1}.$$

Since we have assumed that  $F \in L^q$ , it follows that  $\|f\|_{M_{(\omega)}^{p,q}}$  is finite. This proves (1).

The assertion (2) follows immediately from the general theory of modulation spaces. (See e. g. [19, 20].) The proof is complete.  $\square$

Next we discuss (complex) interpolation properties for modulation spaces. Such properties were carefully investigated in [12] for classical modulation spaces, and thereafter extended in several directions in [15], where interpolation properties for coorbit spaces were established. As a consequence of [15] we have the following proposition.

**Proposition 1.5.** *Assume that  $0 < \theta < 1$ ,  $p_j, q_j \in [1, \infty]$  and that  $\omega_j \in \mathcal{P}(\mathbf{R}^{2n})$  for  $j = 0, 1, 2$  satisfy*

$$\frac{1}{p_0} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q_0} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2} \quad \text{and} \quad \omega_0 = \omega_1^{1-\theta} \omega_2^\theta.$$

Then

$$(\mathcal{M}_{(\omega_1)}^{p_1, q_1}(\mathbf{R}^n), \mathcal{M}_{(\omega_2)}^{p_2, q_2}(\mathbf{R}^n))_{[\theta]} = \mathcal{M}_{(\omega_0)}^{p_0, q_0}(\mathbf{R}^n).$$

Next we recall some facts in Chapter XVIII in [21] concerning pseudo-differential operators. Assume that  $a \in \mathcal{S}'(\mathbf{R}^{2n})$ , and that  $t \in \mathbf{R}$  is fixed. Then the pseudo-differential operator  $a_t(x, D)$  in (0.3) is a linear and continuous operator on  $\mathcal{S}'(\mathbf{R}^n)$ , as remarked in the introduction. For general  $a \in \mathcal{S}'(\mathbf{R}^{2n})$ , the pseudo-differential operator  $a_t(x, D)$  is defined as the continuous operator from  $\mathcal{S}(\mathbf{R}^n)$  to  $\mathcal{S}'(\mathbf{R}^n)$  with distribution kernel

$$(1.3) \quad K_{t,a}(x, y) = (2\pi)^{-n/2} (\mathcal{F}_2^{-1}a)((1-t)x + ty, y-x),$$

Here  $\mathcal{F}_2 F$  is the partial Fourier transform of  $F(x, y) \in \mathcal{S}'(\mathbf{R}^{2n})$  with respect to the  $y$ -variable. This definition makes sense, since the mappings  $\mathcal{F}_2$  and  $F(x, y) \mapsto F((1-t)x + ty, y-x)$  are homeomorphisms on  $\mathcal{S}'(\mathbf{R}^{2n})$ . We also note that this definition of  $a_t(x, D)$  agrees with the operator in (0.3) when  $a \in \mathcal{S}(\mathbf{R}^{2m})$ .

Furthermore, for any  $t \in \mathbf{R}$  fixed, it follows from the kernel theorem by Schwartz that the map  $a \mapsto a_t(x, D)$  is bijective from  $\mathcal{S}'(\mathbf{R}^{2n})$  to  $\mathcal{L}(\mathcal{S}(\mathbf{R}^n), \mathcal{S}'(\mathbf{R}^n))$  (see e. g. [21]).

In particular, if  $a \in \mathcal{S}'(\mathbf{R}^{2m})$  and  $s, t \in \mathbf{R}$ , then there is a unique  $b \in \mathcal{S}'(\mathbf{R}^{2m})$  such that  $a_s(x, D) = b_t(x, D)$ . By straight-forward applications of Fourier's inversion formula, it follows that

$$(1.4) \quad a_s(x, D) = b_t(x, D) \quad \Leftrightarrow \quad b(x, \xi) = e^{i(t-s)\langle D_x, D_\xi \rangle} a(x, \xi).$$

(Cf. Section 18.5 in [21].)

We end this section by recalling some facts on Schatten-von Neumann operators and pseudo-differential operators (cf. the introduction).

For each pairs of Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , the set  $\mathcal{I}_p(\mathcal{H}_1, \mathcal{H}_2)$  is a Banach space which increases with  $p \in [1, \infty]$ , and if  $p < \infty$ , then  $\mathcal{I}_p(\mathcal{H}_1, \mathcal{H}_2)$  is contained in the set of compact operators. Furthermore,  $\mathcal{I}_1(\mathcal{H}_1, \mathcal{H}_2)$ ,  $\mathcal{I}_2(\mathcal{H}_1, \mathcal{H}_2)$  and  $\mathcal{I}_\infty(\mathcal{H}_1, \mathcal{H}_2)$  agree with the set of trace-class operators, Hilbert-Schmidt operators and continuous operators respectively, with the same norms.

Next we discuss complex interpolation properties of Schatten-von Neumann classes. Let  $p, p_1, p_2 \in [1, \infty]$  and let  $0 \leq \theta \leq 1$ . Then it holds

$$(1.5) \quad \mathcal{I}_p = (\mathcal{I}_{p_1}, \mathcal{I}_{p_2})_{[\theta]}, \quad \text{when} \quad \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}.$$

We refer to [?, 27] for a brief discussion of Schatten-von Neumann operators.

For any  $t \in \mathbf{R}$  and  $p \in [1, \infty]$ , let  $s_{t,p}(\omega_1, \omega_2)$  be the set of all  $a \in \mathcal{S}'(\mathbf{R}^{2n})$  such that  $a_t(x, D) \in \mathcal{I}_p(M_{(\omega_1)}^2, M_{(\omega_2)}^2)$ . Also set

$$\|a\|_{s_{t,p}} = \|a\|_{s_{t,p}(\omega_1, \omega_2)} \equiv \|a_t(x, D)\|_{\mathcal{I}_p(M_{(\omega_1)}^2, M_{(\omega_2)}^2)}$$

when  $a_t(x, D)$  is continuous from  $M_{(\omega_1)}^2$  to  $M_{(\omega_2)}^2$ . By using the fact that  $a \mapsto a_t(x, D)$  is a bijective map from  $\mathcal{S}'(\mathbf{R}^{2n})$  to  $\mathcal{L}(\mathcal{S}(\mathbf{R}^n), \mathcal{S}'(\mathbf{R}^n))$ , it follows that the map  $a \mapsto a_t(x, D)$  restricts to an isometric bijection from  $s_{t,p}(\omega_1, \omega_2)$  to  $\mathcal{I}_p(M_{(\omega_1)}^2, M_{(\omega_2)}^2)$ .

Here and in what follows we let  $p' \in [1, \infty]$  denote the conjugate exponent of  $p \in [1, \infty]$ , i. e.  $1/p + 1/p' = 1$ .

**Proposition 1.6.** *Assume that  $p, q_1, q_2 \in [1, \infty]$  are such that  $q_1 \leq \min(p, p')$  and  $q_2 \geq \max(p, p')$ . Also assume that  $\omega_1, \omega_2 \in \mathcal{P}(\mathbf{R}^{2n})$  and  $\omega, \omega_0 \in \mathcal{P}(\mathbf{R}^{4n})$  satisfy*

$$\frac{\omega_2(x - ty, \xi + (1-t)\eta)}{\omega_1(x + (1-t)y, \xi - t\eta)} = \omega(x, \xi, \eta, y)$$

and

$$\omega_0(x, y, \xi, \eta) = \omega((1-t)x + ty, t\xi - (1-t)\eta, \xi + \eta, y - x).$$

Then the following is true:

- (1)  $M_{(\omega)}^{p, q_1}(\mathbf{R}^{2n}) \subseteq s_{t,p}(\omega_1, \omega_2) \subseteq M_{(\omega)}^{p, q_2}(\mathbf{R}^{2n})$ ;
- (2) the operator kernel  $K$  of  $a_t(x, D)$  belongs to  $M_{(\omega_0)}^p(\mathbf{R}^{2n})$  if and only if  $a \in M_{(\omega)}^p(\mathbf{R}^{2n})$  and for some constant  $C$ , which only depends on  $t$  and the involved weight functions, it holds  $\|K\|_{M_{(\omega_0)}^p} = C\|a\|_{M_{(\omega)}^p}$ .

*Proof.* The assertion (1) is a restatement of Theorem 4.13 in [36]. The assertion (2) follows by similar arguments as in the proof of Proposition 4.8 in [36], which we recall here. Let  $\chi, \psi \in \mathcal{S}(\mathbf{R}^{2n})$  be such that

$$\psi(x, y) = \int \chi((1-t)x + ty, \xi) e^{i\langle y-x, \xi \rangle} d\xi.$$

By applying the Fourier inversion formula it follows by straightforward computations that

$$|\mathcal{F}(K\tau_{(x-ty, x+(1-t)y)}\psi)(\xi + (1-t)\eta, -\xi + t\eta)| = |\mathcal{F}(a\tau_{(x, \xi)}\chi)(y, \eta)|.$$

The result now follows by applying the  $L_{(\omega)}^p$  norm on these expressions.  $\square$

## 2. CONTINUITY PROPERTIES OF FOURIER INTEGRAL OPERATORS

In this section we extend Theorem 3.2 in [5] in such way that more general modulation spaces are involved. In these investigations we assume that the phase function  $\varphi$  and the amplitude  $a$  depend on  $x, y \in \mathbf{R}^n$  and  $\zeta \in \mathbf{R}^m$ . For conveniency we use the notation  $X, Y, Z, \dots$  for tripples of the form  $(x, y, \zeta) \in \mathbf{R}^{2n+m}$ . We start to make an appropriate definition of the involved Fourier integral operators.

Assume that  $v \in \mathcal{P}(\mathbf{R}^{2n+m} \times \mathbf{R}^{2n+m})$  is sub-multiplicative and satisfies

$$(2.1) \quad \begin{aligned} v(X, \xi, \eta, z) &= v(\xi, \eta, z), \quad \text{and} \\ v(t \cdot) &\leq C v \quad \xi, \eta \in \mathbf{R}^n \quad z \in \mathbf{R}^m, \end{aligned}$$

for some constant  $C$  which is independent of  $t \in [0, 1]$  (i. e.  $v(X, \xi, \eta, z)$  is constant with respect to  $X \in \mathbf{R}^{2n+m}$ ). For each real-valued  $\varphi \in C^2(\mathbf{R}^{2n+m})$  which satisfies  $\partial^\alpha \varphi \in M_{(v)}^{\infty, 1}$ , and  $a \in \mathcal{S}(\mathbf{R}^{2n+m})$ , it follows that the Fourier integral operator  $f \mapsto \text{Op}_\varphi(a)f$  in (0.2) is well-defined and makes sense as a continuous operator from  $\mathcal{S}(\mathbf{R}^n)$  to  $\mathcal{S}'(\mathbf{R}^n)$ . If  $f, g \in \mathcal{S}(\mathbf{R}^n)$ , then

$$(\text{Op}_\varphi(a)f, g) = (2\pi)^{-n} \int a(X) e^{i\varphi(X)} f(y) \overline{g(x)} dX.$$

In order to extend the definition we reformulate the latter relation in terms of short-time Fourier transforms.

Assume that  $0 \leq \chi, \psi \in C_0^\infty(\mathbf{R}^{2n+m})$  and  $0 \leq \chi_0 \in C_0^\infty(\mathbf{R}^n)$  are such that

$$\|\chi_0\|_{L^1} = \|\chi\|_{L^2} = 1,$$

and that  $X_1 = (x_1, y_1, \zeta_1) \in \mathbf{R}^{2n+m}$  as usual. By straight-forward computations we get

$$\begin{aligned} (\text{Op}_\varphi(a)f, g) &= \int a(X) f(y) \overline{g(x)} e^{i\varphi(X)} dX \\ &= \iint a(X + X_1) \chi(X_1)^2 f(y + y_1) \chi_0(y_1) \overline{g(x + x_1) \chi_0(x_1)} e^{i\psi(X_1)\varphi(X + X_1)} dX dX_1 \end{aligned}$$

If  $\mathcal{F}_{1,2}a$  denotes the partial Fourier transform of  $a(x, y, \zeta)$  with respect to the  $x$  and  $y$  variables, then Parseval's formula gives

$$\begin{aligned} &(\text{Op}_\varphi(a)f, g) \\ &= \iiint F(X, \xi, \eta, \zeta_1) \mathcal{F}(f(y + \cdot) \chi_0)(-\eta) \overline{\mathcal{F}(g(x + \cdot) \chi_0)(\xi)} dX d\xi d\eta d\zeta_1 \\ &= \iiint F(X, \xi, \eta, \zeta_1) (V_{\chi_0} f)(y, -\eta) \overline{(V_{\chi_0} g)(x, \xi)} e^{-i(\langle x, \xi \rangle + \langle y, \eta \rangle)} dX d\xi d\eta d\zeta_1, \\ &= \iiint \left( \int F(X, \xi, \eta, \zeta_1) d\zeta_1 \right) (V_{\chi_0} f)(y, -\eta) \overline{(V_{\chi_0} g)(x, \xi)} e^{-i(\langle x, \xi \rangle + \langle y, \eta \rangle)} dX d\xi d\eta, \end{aligned}$$

where

$$F(X, \xi, \eta, \zeta_1) = \mathcal{F}_{1,2}(e^{i\psi(\cdot, \zeta_1)\varphi(X + (\cdot, \zeta_1))} a(X + (\cdot, \zeta_1)) \chi(\cdot, \zeta_1)^2)(\xi, \eta).$$

By Taylor's formula it follows that

$$\psi(X_1)\varphi(X + X_1) = \psi(X_1)\psi_{1,X}(X_1) + \psi_{2,X}(X_1),$$

where

$$(2.2) \quad \begin{aligned} \psi_{1,X}(X_1) &= \varphi(X) + \langle \varphi'(X), X_1 \rangle \\ \psi_{2,X}(X_1) &= \psi(X_1) \int_0^1 (1-t) \langle \varphi''(X + tX_1) X_1, X_1 \rangle dt. \end{aligned}$$



By inserting these expressions into the definition of  $F(X, \xi, \eta, \zeta_1)$ , and integrating with respect to the  $\zeta_1$ -variable give

$$\begin{aligned} & \int F(X, \xi, \eta, \zeta_1) d\zeta_1 \\ &= \mathcal{F}((e^{i\psi_2, X} \chi)(a(\cdot + X)\chi))(\xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X)) \\ &= \mathcal{H}_{a, \varphi}(X, \xi, \eta), \end{aligned}$$

where

$$(2.3) \quad \begin{aligned} \mathcal{H}_{a, \varphi}(X, \xi, \eta) &= h_X * (\mathcal{F}(a(\cdot + X)\chi))(\xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X)), \\ \text{and } h_X &= (2\pi)^{-n}(\mathcal{F}(e^{i\psi_2, X} \chi)) \end{aligned}$$

Summing up we have proved that

$$(2.4) \quad \begin{aligned} (\text{Op}_\varphi(a)f, g) &= T_{a, \varphi}(f, g) \\ &\equiv \iiint \mathcal{H}_{a, \varphi}(X, \xi, \eta)(V_{\chi_0}f)(y, -\eta) \overline{(V_{\chi_0}g)(x, \xi)} e^{-i(\langle x, \xi \rangle + \langle y, \eta \rangle)} dX d\xi d\eta. \end{aligned}$$

If  $a \in \mathcal{S}'(\mathbf{R}^{2n+m})$ ,  $f, g \in \mathcal{S}(\mathbf{R}^n)$  and that the mapping

$$(X, \xi, \eta) \mapsto \mathcal{H}_{a, \varphi}(X, \xi, \eta)(V_{\chi_0}f)(y, -\eta) \overline{(V_{\chi_0}g)(x, \xi)}$$

belongs to  $L^1(\mathbf{R}^{2n+m} \times \mathbf{R}^{2n})$ , then we still let  $T_{a, \varphi}(f, g)$  be defined as the right-hand side of (3.9). In what follows we use (3.9) to extend the definition of Fourier integral operator with more general amplitudes. Here recall that if for each fixed  $f_0 \in \mathcal{S}$  and  $g_0 \in \mathcal{S}$ , the mappings  $f \mapsto T(f, g_0)$  and  $g \mapsto T(f_0, g)$  are continuous from  $\mathcal{S}$  to  $\mathbf{C}$ , then it follows by Banach-Steinhaus theorem that

$$(f, g) \mapsto T(f, g)$$

is continuous from  $\mathcal{S} \times \mathcal{S}$  to  $\mathbf{C}$ .

**Definition 2.1.** Assume that  $v \in \mathcal{P}(\mathbf{R}^{2n+m} \times \mathbf{R}^{2n+m})$  is submultiplicative and satisfies (2.1),  $\varphi \in C^2(\mathbf{R}^{2n+m})$  is such and that  $\partial^\alpha \varphi \in M_{(v)}^{\infty, 1}$ , and that  $a \in \mathcal{S}'(\mathbf{R}^{2n+m})$  is such that  $f \mapsto T_{a, \varphi}(f, g_0)$  and  $g \mapsto T_{a, \varphi}(f_0, g)$  are well-defined and continuous from  $\mathcal{S}(\mathbf{R}^n)$  to  $\mathbf{C}$ , for each fixed  $f_0, g_0 \in \mathcal{S}(\mathbf{R}^n)$ . Then  $(a, \varphi)$  is called *admissible*, and the Fourier integral operator  $\text{Op}_\varphi(a)$  is the continuous mapping from  $\mathcal{S}(\mathbf{R}^n)$  to  $\mathcal{S}'(\mathbf{R}^n)$  which is defined by the formulas (2.2), (3.7) and (3.9).

Our general continuity results also includes weights which satisfy conditions of the form

$$(2.5) \quad \frac{\omega_2(x, \xi)}{\omega_1(y, -\eta)} \leq C \omega(X, \xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X)),$$

$$X = (x, y, \zeta) \in \mathbf{R}^{2n+m}, \quad \xi, \eta \in \mathbf{R}^n.$$

**Theorem 2.2.** Assume that  $d > 0$ ,  $\omega, v \in \mathcal{P}(\mathbf{R}^{2n+m} \times \mathbf{R}^{2n+m})$  and  $\omega_1, \omega_2 \in \mathcal{P}(\mathbf{R}^{2n})$  are such that (2.1) and (2.5) are fulfilled and that  $v$  is submultiplicative. Also assume that  $\varphi \in C^2(\mathbf{R}^{2n+m})$  is such that  $\partial^\alpha \varphi \in M_{(v)}^{\infty, 1}$ ,  $a \in \mathcal{S}'(\mathbf{R}^{2n+m})$ , and that one of the following conditions hold:

$$(1) \quad |\det(\varphi''_{\zeta, \zeta})| \geq d \text{ and } \|a\| < \infty, \text{ where}$$

$$(2.6) \quad \|a\| = \sup_{X, \xi, \eta} \left( \int |V_\chi a(X, \xi, \eta, z) \omega(X, \xi, \eta, z)| dz \right) < \infty;$$

(2)  $m = n$ ,  $|\det(\varphi''_{x,\zeta})| \geq d$  and  $\|a\| < \infty$ , where

$$(2.7) \quad \|a\| = \sup_{X,\eta,z} \left( \int |V_\chi a(X, \xi, \eta, z) \omega(X, \xi, \eta, z)| d\xi \right);$$

(3)  $m = n$ ,  $|\det(\varphi''_{y,\zeta})| \geq d$  and  $\|a\| < \infty$ , where

$$(2.8) \quad \|a\| = \sup_{X,\xi,z} \left( \int |V_\chi a(X, \xi, \eta, z) \omega(X, \xi, \eta, z)| d\eta \right).$$

Then  $(a, \varphi)$  is admissible, and  $\text{Op}_\varphi(a)$  extends to a linear and continuous operator from  $M^1_{(\omega_1)}(\mathbf{R}^n)$  to  $M^\infty_{(\omega_2)}(\mathbf{R}^n)$ .

Moreover, for some  $C$  which is independent of  $a$  and  $\varphi$  it holds

$$(2.9) \quad \|\text{Op}_\varphi(a)\|_{M^1_{(\omega_1)} \rightarrow M^\infty_{(\omega_2)}} \leq \frac{C\|a\|}{d} \exp(C\|\varphi''\|_{M^\infty_{(v)}}).$$

We need some preparing lemmas for the proof.

**Lemma 2.3.** Assume that  $v(x, \xi) = v(\xi) \in \mathcal{P}(\mathbf{R}^n)$  is submultiplicative and satisfies  $v(t\xi) \leq Cv(\xi)$  for some constant  $C$  which is independent of  $t \in [0, 1]$  and  $\xi \in \mathbf{R}^n$ . Also assume that  $f \in M^{\infty,1}_{(v)}(\mathbf{R}^n)$ ,  $\chi \in C^\infty_0(\mathbf{R}^n)$  and that  $x \in \mathbf{R}^n$ , and let

$$\varphi_{x,j,k}(y) = \chi(y) \int_0^1 (1-t)f(x+ty)y_j y_k dt.$$

Then there is a constant  $C$  and a function  $g \in M^1_{(v)}(\mathbf{R}^n)$  such that  $\|g\|_{M^1_{(v)}} \leq C\|f\|_{M^{\infty,1}_{(v)}}$  and  $|\mathcal{F}(\varphi_{x,j,k})(\xi)| \leq \widehat{g}(\xi)$ .

*Proof.* We first prove the assertion when  $\chi$  is replaced by  $\psi_0(y) = e^{-2|y|^2}$ . For conveniency we let

$$\psi_1(y) = e^{-|y|^2}, \quad \text{and} \quad \psi_2(y) = e^{|y|^2},$$

and

$$H_{\infty,f}(\xi) \equiv \sup_x |\mathcal{F}(f \psi_1(\cdot - x))(\xi)|.$$

We claim that  $g$ , defined by

$$(2.10) \quad \widehat{g}(\xi) = \int_0^1 \int (1-t)H_{\infty,f}(\eta)e^{-|\xi-t\eta|^2/16} d\eta dt,$$

fulfills the required properties.

In fact, by applying  $M^1_{(v)}$  norm on  $g$  and using Minkowski's inequality and Remark 1.3 (6), we obtain

$$\begin{aligned} \|g\|_{M^1_{(v)}} &= \left\| \int_0^1 \int (1-t)H_{\infty,f}(\eta)e^{-|\xi-t\eta|^2/16} d\eta dt \right\|_{M^1_{(v)}} \\ &\leq \int_0^1 \int (1-t)H_{\infty,f}(\eta) \|e^{-|\cdot-t\eta|^2/16}\|_{M^1_{(v)}} d\eta dt \\ &\leq C_1 \int_0^1 \int (1-t)H_{\infty,f}(\eta) \|e^{-|\cdot|^2/16}\|_{M^1_{(v)}} v(t\eta) d\eta dt \\ &\leq C_2 \int_0^1 \int (1-t)H_{\infty,f}(\eta) v(\eta) \|e^{-|\cdot|^2/16}\|_{M^1_{(v)}} d\eta dt \\ &= C_3 \|H_{\infty,f}v\|_{L^1} = C_3 \|f\|_{M^{\infty,1}_{(v)}}. \end{aligned}$$

In order to prove that  $|\mathcal{F}(\varphi_{x,j,k})(\xi)| \leq g(\xi)$ , we let  $\psi(y) = \psi_{j,k}(y) = y_j y_k \psi_0(y)$ . Then

$$\varphi_{x,j,k}(y) = \psi(y) \int_0^1 (1-t) f(x+ty) dt.$$

By a change of variables we obtain

$$\begin{aligned} |\mathcal{F}(\varphi_{x,j,k})(\xi)| &= \left| \int_0^1 (1-t) \left( \int f(x+ty) \psi(y) e^{-i\langle y, \xi \rangle} dy \right) dt \right| \\ (2.11) \quad &= \left| \int_0^1 t^{-n} (1-t) \mathcal{F}(f \psi((\cdot - x)/t))(\xi/t) e^{i\langle x, \xi \rangle/t} dt \right| \\ &\leq \int_0^1 t^{-n} (1-t) \sup_{x \in \mathbf{R}^n} |\mathcal{F}(f \psi((\cdot - x)/t))(\xi/t)| dt. \end{aligned}$$

We need to estimate the right-hand side. By straight-forward computations we get

$$\begin{aligned} &|\mathcal{F}(f \psi((\cdot - x)/t))(\xi)| \\ &\leq (2\pi)^{-n/2} (|\mathcal{F}(f \psi_1(\cdot - x))| * |\mathcal{F}(\psi((\cdot - x)/t) \psi_2(\cdot - x))|)(\xi) \\ &= (2\pi)^{-n/2} (|\mathcal{F}(f \psi_1(\cdot - x))| * |\mathcal{F}(\psi(\cdot/t) \psi_2)|)(\xi) \end{aligned}$$

where the convolutions should be taken with respect to the  $\xi$ -variable only. This implies that

$$(2.12) \quad |\mathcal{F}(f \psi((\cdot - x)/t))(\xi)| \leq (2\pi)^{-n/2} (H_{\infty, f} * |\mathcal{F}(\psi(\cdot/t) \psi_2)|)(\xi)$$

In order to estimate the latter Fourier transform we note that

$$(2.13) \quad |\mathcal{F}(\psi(\cdot/t) \psi_2)| = |\partial_j \partial_k \mathcal{F}(\psi_0(\cdot/t) \psi_2)|.$$

Since  $\psi_0$  and  $\psi_2$  are Gauss functions and  $0 \leq t \leq 1$ , a straight-forward computation gives

$$(2.14) \quad \mathcal{F}(\psi_0(\cdot/t) \psi_2)(\xi) = \pi^{n/2} t^n (2-t^2)^{-n/2} e^{-t^2 |\xi|^2 / (4(2-t^2))}.$$

A combination of (2.13) and (2.14) therefore give

$$(2.15) \quad |\mathcal{F}(\psi(\cdot/t) \psi_2)(\xi)| \leq C t^n e^{-t^2 |\xi|^2 / 16},$$

for some constant  $C$  which is independent of  $t \in [0, 1]$ . The assertion now follows by combining (2.11), (2.12) and (2.15).

In order to prove the result for general  $\chi \in C_0^\infty(\mathbf{R}^n)$  we set

$$h_{x,j,h}(y) = \psi_0(y) \int_0^1 (1-t) f(x+ty) y_j y_k dt,$$

and we observe that the result is already proved when  $\varphi_{x,j,k}$  is replaced by  $h_{x,j,h}$  and moreover  $\varphi_{x,j,k} = \chi_1 h_{x,j,h}$ , for some  $\chi_1 \in C_0^\infty(\mathbf{R}^n)$ . Hence if  $g_0$  is given as the right-hand side of (2.10), the first part of the proof shows that

$$|\mathcal{F}(\varphi_{x,j,k})(\xi)| = |\mathcal{F}(\chi_1 h_{x,j,h})(\xi)| \leq (2\pi)^{-n/2} |\widehat{\chi_1}| * g - 0(\xi) \equiv g(\xi).$$

Moreover  $\|g_0\|_{M_{(v)}^1} \leq C \|f\|_{M_{(v)}^{\infty,1}}$ .

Since  $M_{(v)}^1 * L_{(v)}^1 \subseteq M_{(v)}^1$ , we get for some positive constants  $C, C_1$

$$\|g\|_{M_{(v)}^1} \leq C \|\widehat{\chi_1}\|_{L_{(v)}^1} \|g_0\|_{M_{(v)}^1} \leq C_1 \|f\|_{M_{(v)}^{\infty,1}},$$

which proves the result □

As a consequence of Lemma 2.3 we have the following result.

**Lemma 2.4.** *Assume that  $v(x, \xi) = v(\xi) \in \mathcal{P}(\mathbf{R}^n)$  is submultiplicative and satisfies  $v(t\xi) \leq Cv(\xi)$  for some constant  $C$  which is independent of  $t \in [0, 1]$  and  $\xi \in \mathbf{R}^n$ . Also assume that  $f_{j,k} \in M_{(v)}^{\infty,1}(\mathbf{R}^n)$  for  $j, k = 1, \dots, n$ ,  $\chi \in C_0^\infty(\mathbf{R}^n)$  and that  $x \in \mathbf{R}^n$ , and let*

$$\varphi_x(y) = \sum_{j,k=1,\dots,n} \varphi_{x,j,k}(y), \quad \text{where} \quad \varphi_{x,j,k}(y) = \chi(y) \int_0^1 (1-t) f_{j,k}(x+ty) y_j y_k dt.$$

Then there is a constant  $C$  and a function  $\Psi \in M_{(v)}^1(\mathbf{R}^n)$  such that

$$\|\Psi\|_{M_{(v)}^1} \leq \exp(C \sup_{j,k} \|f_{j,k}\|_{M_{(v)}^{\infty,1}})$$

and

$$(2.16) \quad |\mathcal{F}(\exp(i\varphi_x)(\xi))| \leq (2\pi)^{n/2} \delta_0 + \widehat{\Psi}(\xi).$$

*Proof.* By Lemma 2.3, we may find a function  $g \in M_{(v)}^1$  and a constant  $C$  such that

$$|\widehat{\varphi_x}(\xi)| \leq \widehat{g}(\xi), \quad \|g\|_{M_{(v)}^1} \leq C \sup_{j,k} (\|f_{j,k}\|_{M_{(v)}^{\infty,1}}).$$

Set

$$\Phi_{0,x} = (2\pi)^{n/2} \delta_0, \quad \Phi_{l,x} = |\mathcal{F}(\varphi_x)| * \dots * |\mathcal{F}(\varphi_x)|, \quad l \geq 1$$

$$\Upsilon_0 = (2\pi)^{n/2} \delta_0, \quad \Upsilon_l = g * \dots * g, \quad l \geq 1,$$

with  $l$  factors in the convolutions. Then by Taylor series expanding, there is a constant  $C$  such that

$$|\mathcal{F}(\exp(i\varphi_x(\cdot))(\xi))| \leq \sum_{l=0}^{\infty} C^l \Phi_{l,x} / l! \leq \sum_{l=0}^{\infty} C^l \Upsilon_l / l!$$

Hence, if we set

$$\Psi = \sum_{l=1}^{\infty} C^l \Upsilon_l / l!,$$

it follows that (2.16) holds.

Furthermore, since  $v$  is submultiplicative we have

$$\|\Upsilon_l\|_{M_{(v)}^1} = \|g * \dots * g\|_{M_{(v)}^1} \leq (C_1 \|g\|_{M_{(v)}^1})^l,$$

for some constant  $C_1$ . This gives

$$\begin{aligned} \|\Psi\|_{M_{(v)}^1} &\leq \sum_{l=1}^{\infty} \|\Upsilon_l\|_{M_{(v)}^1} \\ &\leq \sum_{l=1}^{\infty} (C_1 \|g\|_{M_{(v)}^1})^l \leq \sum_{l=1}^{\infty} (C_2 \sup_{j,k} (\|f_{j,k}\|_{M_{(v)}^{\infty,1}}))^l \leq \exp(C_2 \sup_{j,k} \|f_{j,k}\|_{M_{(v)}^{\infty,1}}), \end{aligned}$$

for some constants  $C_1$  and  $C_2$ . This proves the assertion.  $\square$

*Proof of Theorem 2.2.* We only prove the result when (1) is fulfilled. The other cases follow by similar arguments and are left for the reader.

Since

$$|\mathcal{F}(e^{i\Psi_{2,X}} \chi)| \leq (2\pi)^{-n+m/2} |\mathcal{F}(e^{i\Psi_{2,X}} * |\widehat{\chi}|)|, \quad |\mathcal{F}(a(\cdot + X)\chi)| = |V_\chi a(X, \cdot)|,$$

(3.7), and Lemma 2.4 give

$$|\mathcal{H}_{a,\varphi}(X, \xi, \eta)| \leq C(G * |V_\chi a(X, \cdot)|)(\xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X)),$$

for some  $G \in L^1_{(v)}$  which satisfies  $\|G\|_{L^1_{(v)}} \leq C \exp(C\|\varphi''\|_{M_{(v)}^{\infty,1}})$ . By combining this with (2.5) and letting

$$(2.17) \quad \begin{aligned} E_{a,\omega}(\xi, \eta, z) &= \sup_X |V_\chi a(X, \xi, \eta, z))\omega(X, \xi, \eta, z)| \\ F_1(x, \xi) &= |V_{\chi_0} f(x, \xi)\omega_1(x, \xi)|, \\ F_2(x, \xi) &= |V_{\chi_0} g(x, \xi)/\omega_2(x, \xi)|, \end{aligned}$$

we get

$$\begin{aligned} & \iiint |\mathcal{H}_{a,\varphi}(X, \xi, \eta)(V_{\chi_0} f)(y, -\eta)(V_{\chi_0} g)(x, \xi)| dX d\xi d\eta \\ & \leq C_1 \iiint (G * |V_\chi a(X, \cdot)|)(\xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X)) \times \\ & \quad |(V_{\chi_0} f)(y, -\eta)(V_{\chi_0} g)(x, \xi)| dX d\xi d\eta \\ & \leq C_2 \iiint (G * E_{a,\omega}(\xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X)) \times \\ & \quad F_1(y, -\eta)F_2(x, \xi) dX d\xi d\eta \\ & \leq C_3 \|G\|_{L^1_{(v)}} \iiint (E_{a,\omega}(\xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X)) \times \\ & \quad F_1(y, -\eta)F_2(x, \xi) dX d\xi d\eta. \end{aligned}$$

Summing up we have proved that

$$(2.18) \quad \begin{aligned} & |(\text{Op}_\varphi(a)f, g)| \leq C \exp(C\|\varphi''\|_{M_{(v)}^{\infty,1}}) \times \\ & \quad \times \iiint (E_{a,\omega}(\xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X))F_1(y, -\eta)F_2(x, \xi) dX d\xi d\eta. \end{aligned}$$

By taking  $x, y, -\varphi'_\zeta(X), \xi, \eta$  as new variables of integration, and using the fact that  $|\det(\varphi''_{\zeta,\zeta})| \geq d$  we get

$$\begin{aligned} & \iiint |\mathcal{H}_{a,\varphi}(X, \xi, \eta)(V_{\chi_0} f)(y, -\eta)(V_{\chi_0} g)(x, \xi)| dX d\xi d\eta \\ & \leq \frac{C_3}{d} \|\varphi''\|_{M_{(v)}^{\infty,1}} \iiint \left( \int E_{a,\omega}(\xi - \varphi'_x(X), \eta - \varphi'_y(X), z) dz \right) \times \\ & \quad |(V_{\chi_0} f)(y, -\eta)\omega_1(y, -\eta)(V_{\chi_0} g)(x, \xi)/\omega_2(x, \xi)| dx dy d\xi d\eta \\ & \leq \frac{C_3 \|a\|}{d} \exp(C\|\varphi''\|_{M_{(v)}^{\infty,1}}) \iiint |(V_{\chi_0} f)(y, -\eta)\omega_1(y, -\eta)(V_{\chi_0} g)(x, \xi)/\omega_2(x, \xi)| dx dy d\xi d\eta \\ & \quad = \frac{C_3 \|a\|}{d} \exp(C\|\varphi''\|_{M_{(v)}^{\infty,1}}) \|f\|_{M_{(\omega_1)}^1} \|g\|_{M_{(1/\omega_2)}^1}. \end{aligned}$$

This proves that (2.9) holds, and the result follows.  $\square$

Next we consider Fourier integral operators with symbols in  $M_{(\omega)}^{\infty,1}(\mathbf{R}^{2n+m})$ . The following result generalizes Theorem 3.2 in [6].

**Theorem 2.5.** *Assume that  $1 < p < \infty$ ,  $d > 0$ ,  $\omega, v \in \mathcal{P}(\mathbf{R}^{2n+m} \times \mathbf{R}^{2n+m})$  and  $\omega_1, \omega_2 \in \mathcal{P}(\mathbf{R}^{2n})$  are such that (2.1) and (2.5) are fulfilled and that  $v$  is submultiplicative. Also assume that  $\varphi \in C^2(\mathbf{R}^{2n+m})$  is such that  $\partial^\alpha \varphi \in M_{(v)}^{\infty,1}$  for  $|\alpha| = 2$  and (0.4) are fulfilled. Then the following is true:*

- (1) the map  $a \mapsto \text{Op}_\varphi(a)$  from  $\mathcal{S}(\mathbf{R}^{2n+m})$  to  $\mathcal{L}(\mathcal{S}(\mathbf{R}^n), \mathcal{S}'(\mathbf{R}^n))$  extends uniquely to a continuous map from  $M_{(\omega)}^{\infty,1}(\mathbf{R}^{2n+m})$  to  $\mathcal{L}(\mathcal{S}(\mathbf{R}^n), \mathcal{S}'(\mathbf{R}^n))$ ;
- (2) if  $a \in M_{(\omega)}^{\infty,1}(\mathbf{R}^{2n+m})$ , then the map  $\text{Op}_\varphi(a)$  from  $\mathcal{S}(\mathbf{R}^n)$  to  $\mathcal{S}'(\mathbf{R}^n)$  is uniquely extendable to a continuous operator from  $M_{(\omega_1)}^p(\mathbf{R}^n)$  to  $M_{(\omega_2)}^p(\mathbf{R}^n)$ . Moreover, for some constant  $C$  it holds

$$(2.19) \quad \|\text{Op}_\varphi(a)\|_{M_{(\omega_1)}^p \rightarrow M_{(\omega_2)}^p} \leq C d^{-1} \|a\|_{M_{(\omega)}^{\infty,1}} \exp(C \|\varphi''\|_{M_{(v)}^{\infty,1}}).$$

*Proof.* We shall mainly follow the proof of Theorem 3.2 in [6]. First assume that  $a \in C_0^\infty(\mathbf{R}^{2n+m})$ . Let  $f, g \in \mathcal{S}(\mathbf{R}^n)$ . Then it follows that  $\text{Op}_\varphi(a)$  makes sense as a continuous operator from  $\mathcal{S}$  to  $\mathcal{S}'$ . By letting

$$C_\varphi = C \exp(C \|\varphi''\|_{M_{(v)}^{\infty,1}}),$$

where  $C$  is the same as in (2.18), it follows from (2.17), (2.18) and Hölder's inequality that

$$(2.20) \quad |(\text{Op}_\varphi(a)f, g)| \leq C_\varphi J_1 \cdot J_2,$$

where

$$J_1 = \left( \iiint (E_{a,\omega}(\xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X)) F_1(y, -\eta)^p dX d\xi d\eta) \right)^{1/p}$$

$$J_2 = \left( \iiint (E_{a,\omega}(\xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X)) F_2(y, -\eta)^{p'} dX d\xi d\eta) \right)^{1/p'}.$$

We have to estimate  $J_1$  and  $J_2$ . By taking  $z = \varphi'_3(X)$ ,  $\zeta_0 = \varphi'_2(X)$ ,  $y$ ,  $\xi$  and  $\eta$  as new variables of integrations, and using (0.4), it follows that

$$\begin{aligned} J_1 &\leq \left( d^{-1} \iiint (E_{a,\omega}(\xi - \varphi'_x(X), \eta - \zeta_0, z) F_1(y, -\eta)^p dy dz d\xi d\eta d\zeta_0) \right)^{1/p} \\ &= \left( d^{-1} \iiint (E_{a,\omega}(\xi, \zeta_0, z) F_1(y, -\eta)^p dy dz d\xi d\eta d\zeta_0) \right)^{1/p} \\ &= d^{-1/p} \|E_{a,\omega}\|_{L^1}^{1/p} \|F_1\|_{L^p}. \end{aligned}$$

Hence

$$(2.21) \quad J_1 \leq d^{-1/p} \|a\|_{M_{(\omega)}^{\infty,1}}^{1/p} \|f\|_{M_{(\omega_1)}^p}.$$

If we instead take  $x$ ,  $y_0 = \varphi'_3(X)$ ,  $\xi$ ,  $\eta$  and  $\zeta_0 = \varphi'_1(X)$  as new variables of integrations, it follows by similar arguments that

$$(2.21)' \quad J_2 \leq d^{-1/p'} \|a\|_{M_{(\omega)}^{\infty,1}}^{1/p'} \|g\|_{M_{(1/\omega_2)}^{p'}}.$$

A combination of (3.10), (3.11) and (3.11)' now give

$$|(\text{Op}_\varphi(a)f, g)| \leq C d^{-1} \|a\|_{M_{(\omega)}^{\infty,1}} \|f\|_{M_{(\omega_1)}^p} \|g\|_{M_{(1/\omega_2)}^{p'}} \exp(\|\varphi''\|_{M_{(v)}^{\infty,1}}),$$

which proves (2.19), and the result follows in this case.

Since  $\mathcal{S}$  is dense in  $M_{(\omega_1)}^p$  and  $M_{(1/\omega_2)}^{p'}$ , the result also holds for  $a \in C_0^\infty$  and  $f \in M_{(\omega_1)}^p$ . Hence it follows by Hahn-Banach's theorem that the asserted extension of the map  $a \mapsto \text{Op}_\varphi(a)$  exists.

It remains to prove that this extension is unique. Therefore assume that  $a \in M_{(\omega)}^{\infty,1}$  is arbitrary, and take a sequence  $a_j \in C_0^\infty$  for  $j = 1, 2, \dots$  which converges to  $a$  with respect to the narrow convergence (cf. [28, 36]). Then  $E_{a_j, \omega}$  converges to

$E_{a,\omega}$  in  $L^1$  as  $j$  turns to infinity. By (2.2)–(3.9) and the arguments at the above, it follows from Lebesgue's theorem that

$$(\text{Op}_\varphi(a_j)f, g) \rightarrow (\text{Op}_\varphi(a)f, g)$$

as  $j$  turns to infinity. This proves the uniqueness, and the result follows.  $\square$

*Remark 2.6.* Assume that  $a \in M^{\infty,1}(\mathbf{R}^{2n+m})$  and that the assumptions on  $\varphi$  in Definition 2.1 is fulfilled with  $v \equiv 1$ . Also assume that  $\kappa \in C_0^\infty(\mathbf{R}^m)$  satisfies  $\kappa(0) = 1$ . Then it is proved in [5], the Fourier integral operator

### 3. SCHATTEN-VON NEUMANN PROPERTIES OF FOURIER INTEGRAL OPERATORS

In this section we discuss Schatten-von Neumann operators for Fourier integral operators with symbols in  $M_{(\omega)}^{p,q}(\mathbf{R}^{2n})$  and phase functions in  $M_{(v)}^{\infty,1}(\mathbf{R}^{3n})$ , for appropriate  $\omega$  and  $v$ . In these investigations we assume that the phase functions depend on  $x, y, \xi \in \mathbf{R}^n$  and that the symbols are independent of the  $y$  variable, and for conveniency we use the notation  $X, Y, Z, \dots$  for tripples of the form  $(x, y, \xi) \in \mathbf{R}^{3n}$ .

In order to establish a weighted version of Theorem 2.5 in [?] we list some conditions for the weight and phase functions. In what follows we assume that  $\varphi \in C^2(\mathbf{R}^{3n})$ ,  $\omega_0, \omega \in \mathcal{P}(\mathbf{R}^{4n})$ ,  $v_1 \in \mathcal{P}(\mathbf{R}^n)$ ,  $v_2 \in \mathcal{P}(\mathbf{R}^{2n})$ ,  $v \in \mathcal{P}(\mathbf{R}^{6n})$  and  $s_j, t_j \in \mathbf{R}$  for  $j = 1, 2$  satisfy

$$(3.1) \quad |\det(\varphi''_{y,\xi}(x, y, \xi))| \geq d$$

$$(3.2) \quad \omega_0(x, y, \xi, \varphi'_y(x, y, \eta)) = \omega(x, \eta, \xi - \varphi'_x(x, y, \eta), -\varphi'_\eta(x, y, \eta))$$

and

$$(3.3) \quad \begin{aligned} \omega_0(x, y, \xi, \eta + \zeta) &\leq \omega_0(x, y, \xi, \eta)v_1(\zeta), & x, y, \xi, \eta, \zeta \in \mathbf{R}^n \\ \omega(x, \eta, \xi_1 + \xi_2, y_1 + y_2) &\leq \omega(x, \eta, \xi_1, y_1)v_2(\xi_2, y_2), & x, y_1, y_2, \xi_1, \xi_2, \eta \in \mathbf{R}^n \\ v(x, y, \zeta, \xi, \eta, z) &= v_1(\eta)v_2(\xi, z), & x, y, z, \xi, \eta, \zeta \in \mathbf{R}^n. \end{aligned}$$

**Theorem 3.1.** Assume that  $p \in [1, \infty]$ ,  $d > 0$ ,  $v \in \mathcal{P}(\mathbf{R}^{6n})$  is submultiplicative,  $\omega_0, \omega \in \mathcal{P}(\mathbf{R}^{4n})$  and that  $\varphi \in C(\mathbf{R}^{3n})$  are such that  $\varphi$  is real-valued,  $\partial^\alpha \varphi \in M_{(v)}^{\infty,1}$  for  $|\alpha| = 2$  and (3.1)–(3.3) are fulfilled. Then the map

$$a \mapsto K_{a,\varphi}(x, y) \equiv \int a(x, \xi) e^{i\varphi(x, y, \xi)} d\xi,$$

from  $\mathcal{S}(\mathbf{R}^{2n})$  to  $\mathcal{S}'(\mathbf{R}^{2n})$  extends uniquely to a continuous map from  $M_{(\omega)}^p(\mathbf{R}^{2n})$  to  $M_{(\omega_0)}^p(\mathbf{R}^{2n})$ .

For the proof we need the following lemma.

*Proof of Theorem 3.1.* First assume that  $p = 1$ , and let  $\chi \in C_0^\infty(\mathbf{R}^n)$  be such that  $\int \chi_1 dx = 1$ ,  $\chi = \chi_2 = \chi_1 \otimes \chi_1$ ,  $\chi_3 = \chi_1 \otimes \chi_1 \otimes \chi_1$  and  $\psi \in C_0^\infty(\mathbf{R}^{3n})$  be such that  $\psi = 1$  in  $\text{supp } \chi_3$ . We also let  $X_1 = (x_1, y_1, \zeta_1)$ ,  $X = (x, y, \zeta)$ , and consider the modulus of short-time Fourier transform of the distribution kernel of  $K_{a,\varphi}$ , i. e.

$$\begin{aligned} I_a(x, y, \xi, \eta) &= |\mathcal{F}(K_{a,\varphi} \chi_2(\cdot - (x, y)))(\xi, \eta)| \\ &= \left| \int a(x_1, \zeta_1) e^{i\varphi(X_1)} \chi_2(x_1 - x, y_1 - y) e^{-i(\langle x_1, \xi \rangle + \langle y_1, \eta \rangle)} dX_1 \right|, \\ &= \left| \iint a(x_1, \zeta_1) e^{i\varphi(X_1)} \chi_3(X_1 - X) e^{-i(\langle x_1, \xi \rangle + \langle y_1, \eta \rangle)} dX_1 d\zeta \right|, \end{aligned}$$

and note that the  $L^1_{(\omega_0)}$ -norm of  $I_a$  is equivalent to the  $M^1_{(\omega_0)}$ -norm of  $K_{a,\varphi}$  in view of Remark 1.3. By a change of variables it follows that

$$I_a(x, y, \xi, \eta) = \left| \iint a(x_1 + x, \zeta_1 + \zeta) e^{i\varphi(X_1+X)} \chi_3(X_1) e^{-i(\langle x_1, \xi \rangle + \langle y_1, \eta \rangle)} dX_1 d\zeta \right|$$

In a similar way as in Section 2 we let  $\psi_{1,X}$  and  $\psi_{2,X}$  be defined as in (2.2). By letting  $a_1(x, y, \xi) = a(x, \xi)$ , an application of Taylor formula on  $\varphi$  gives

$$\begin{aligned} I_a(x, y, \xi, \eta) &= \left| \iint a(x_1 + x, \zeta_1 + \zeta) e^{i\psi_{2,X}(X_1)} \chi_3(X_1) e^{-i(\langle x_1, \xi \rangle + \langle y_1, \eta \rangle - \psi_{1,X}(X_1))} dX_1 d\zeta \right| \\ &= \left| \int e^{i\varphi(X)} \mathcal{F}(a(\cdot + x, \cdot + \zeta) \chi_3 e^{i\psi_{2,X}})(\xi - \varphi'_1(X), \eta - \varphi'_2(X), -\varphi'_3(X)) d\zeta \right| \\ &\leq \int (|\mathcal{F}(a_1 \chi_3(\cdot - X))| * |\mathcal{F}(e^{i\psi_{2,X}})|(\xi - \varphi'_1(X), \eta - \varphi'_2(X), -\varphi'_3(X)) d\zeta \end{aligned}$$

This implies that

$$(3.4) \quad I_a(x, y, \xi, \eta) \leq \sum_{k=0}^{\infty} I_{a,k}(x, y, \xi, \eta) / k!,$$

where

$$\begin{aligned} I_{a,0}(x, y, \xi, \eta) &\equiv \int (|\mathcal{F}(a_1 \chi_3(\cdot - X))|(\xi - \varphi'_1(X), \eta - \varphi'_2(X), -\varphi'_3(X)) d\zeta \\ I_{a,k}(x, y, \xi, \eta) &\equiv \int (|\mathcal{F}(a_1 \chi_3(\cdot - X))| * \Phi_{k,X}(\xi - \varphi'_1(X), \eta - \varphi'_2(X), -\varphi'_3(X)) d\zeta, \\ \Phi_{k,X} &\equiv |\mathcal{F}(\psi_{2,X})| * \cdots * |\mathcal{F}(\psi_{2,X})|, \quad k \geq 1. \end{aligned}$$

Here the number of factors in the latter convolutions is equal to  $k$ .

We need to estimate the  $L^1_{(\omega_0)}$  norm of  $I_{a,k}(x, y, \xi, \eta)$ , and start to consider the case  $k = 0$ .

Next we consider  $I_{a,k}(x, y, \xi, \eta)$  when  $k \geq 1$ . An application of Lemma 3.1 shows that there is a function  $G$  such that  $|\mathcal{F}(\psi_{2,X})| \leq G$  and  $\|G\|_{L^1_{(v)}} \leq C\|\varphi''\|_{M^{\infty,1}_{(v)}}$  for some constant  $C > 0$ . Hence if  $\Upsilon_k \equiv G * \cdots * G$  with  $k$  factors of  $G$  in the convolution, then it follows that  $|\Phi_{k,X}| \leq \Upsilon_k$  and that  $\|\Upsilon_k\|_{L^1_{(v)}} \leq C^k \|\varphi''\|_{M^{\infty,1}_{(v)}}^k$ , where the latter inequality follows from the fact that  $v$  is submultiplicative.

By letting  $\xi_0 = \xi - \varphi'_1(X)$ ,  $y_0 = -\varphi'_3(X)$  and

$$J_k(x, y, \xi, \eta, z) = |\mathcal{F}(a_1 \chi_3(\cdot - X))| * \Upsilon_k(\xi_0, \eta - \varphi'_2(X), y_0),$$

we therefore get

$$I_{a,k}(x, y, \xi, \eta) \leq \int J_k(x, y, \xi, \eta, \zeta) d\zeta$$

and

$$\begin{aligned} J_k(x, y, \xi, \eta, \zeta) &\leq \iiint |V_\chi a(x, \zeta, \xi_0 - \xi_1, y_0 - y_1)| \widehat{\chi}(\eta - \varphi'_2(X) - \eta_1) |\Upsilon_k(\xi_1, \eta_1, y_1)| d\xi_1 d\eta_1 dy_1. \end{aligned}$$



By (3.3) we have for some constant  $C$  that

$$\begin{aligned}\omega_0(x, y, \xi, \eta) &\leq C\omega_0(x, y, \xi, \varphi'_2(X))v_1(\eta - \varphi_2(X) - \eta_1)v_1(\eta_1) \\ \omega_0(x, y, \xi, \varphi_2(X)) &= \omega(x, \zeta, \xi_0, y_0) \\ &\leq C\omega(x, \zeta, \xi_0 - \xi_1, y_0 - y_1)v_2(\xi_1, y_1),\end{aligned}$$

which implies that

$$\omega_0(x, y, \xi, \eta) \leq C^2\omega(x, \zeta, \xi_0 - \xi_1, y_0 - y_1)v_1(\eta - \varphi_2(X) - \eta_1)v(\xi_1, \eta_1, y_1)$$

Hence if

$$\begin{aligned}F(x, \zeta, \xi, z) &= |V_\chi a(x, \zeta, \xi, z)\omega(x, \zeta, \xi, z)| \\ G(\eta) &= |v_1(\eta)\widehat{\chi}(\eta)| \\ H_k(\eta) &= \Upsilon_k(\xi, \eta, y)v_1(\eta)v_2(\xi, y) = \Upsilon_k(\xi, \eta, y)v(\xi, \eta, y),\end{aligned}$$

then we get

$$\begin{aligned}J_k(x, y, \xi, \eta, \zeta)\omega_0(x, y, \xi, \eta) \\ \leq C \iiint F(x, \zeta, \xi_0 - \xi_1, y_0 - y_1)G(\eta - \varphi'_2(X) - \eta_1)H_k(\xi_1, \eta_1, y_1) d\xi_1 d\eta_1 dy_1.\end{aligned}$$

By applying the  $L^1$ -norm on the latter estimate we get

$$\begin{aligned}\|I_{a,k}\|_{L^1_{(\omega_0)}} &\leq C_1 \int \|J_k(\cdot, \zeta)\omega_0\|_{L^1} d\zeta \\ &\leq C_2 \|G\|_{L^1} \|H_k\|_{L^1} \iiint F(x, \zeta, \xi, -\varphi'_3(X)) dx dy d\xi d\zeta \\ &\leq C_3 \|\varphi''\|_{M_{(v)}^{\infty,1}}^k \iiint F(x, \zeta, \xi, -\varphi'_3(X)) dx dy d\xi d\zeta,\end{aligned}$$

for some constants  $C_1, \dots, C_3$ . Hence, by taking  $(x, -\varphi'_3(X), \xi, \zeta)$  as new variables of integration, and using (3.1) we get

$$\|I_{a,k}\|_{L^1_{(\omega_0)}} \leq C d^{-1} \|F\|_{L^1} \|\varphi''\|_{M_{(v)}^{\infty,1}}^k.$$

Hence, by applying the  $L^1_{(\omega_0)}$  norm on the latter estimate we get

$$\begin{aligned}\|K_{a,\varphi}\|_{M^1_{(\omega_0)}} &\leq \|I_a\|_{L^1} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \iiint (|\mathcal{F}(a \otimes \chi)| * |\Upsilon_k|)(\xi - \partial_x \varphi(X_0), -\partial_\zeta \varphi(X_0), \eta - \partial_y \varphi(X_0)) dy d\xi d\eta \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \iiint (|\mathcal{F}(a \otimes \chi)| * |\Upsilon_k|)(\xi, -\partial_\zeta \varphi(X_0), \eta) dy d\xi d\eta \\ &\leq d^{-1} \sum_{k=0}^{\infty} \frac{1}{k!} \iiint (|\mathcal{F}(a \otimes \chi)| * |\Upsilon_k|)(\xi, x, \eta) dx d\xi d\eta \\ &\leq C d^{-1} \sum_{k=0}^{\infty} \frac{1}{k!} \|a\|_{M^1} (C \|\varphi''\|_{M^{\infty,1}})^k = C d^{-1} \|a\|_{M^1} \exp(C \|\varphi''\|_{M^{\infty,1}}),\end{aligned}$$

where the second inequality follows from (3.1) and taking  $x = -\partial_\zeta \varphi(X_0)$  as new variable of integration in the  $y$ -direction. This proves the assertion in this case.

For general  $a \in M^1$ , the asserted continuity now follows by applying Proposition 1.4 in a way similar as in the proof of Proposition ?? . We leave the details for the reader.

Next we consider the case  $p = \infty$ . Assume that  $a \in M_{(\omega)}^1(\mathbf{R}^{2n})$  and  $b \in M_{(1/\omega)}^1(\mathbf{R}^{2n})$ , and let  $\tilde{\varphi}(x, y, \xi) = -\varphi(x, \xi, y)$ . Then (3.1) also holds when  $\varphi$  is replaced by  $\tilde{\varphi}$ . Hence, the first part of the proof shows that  $K_{b, \tilde{\varphi}} \in M_{(1/\omega_0)}^1$ . Furthermore, by straight-forward computations we have

$$(3.5) \quad (K_{a, \varphi}, b) = (a, K_{b, \tilde{\varphi}}).$$

In view of Proposition 1.1 (3), it follows that the right-hand side in (3.5) makes sense if, more generally,  $a$  is an arbitrary element in  $M_{(\omega)}^\infty(\mathbf{R}^{2n})$ , and then

$$|(a, K_{b, \tilde{\varphi}})| \leq C d^{-1} \|a\|_{M_{(\omega)}^\infty} \|b\|_{M_{(1/\omega)}^1} \exp(C \|\varphi''\|_{M_{(v)}^{\infty, 1}}),$$

for some constant  $C$  which is independent of  $d$ ,  $a \in M^\infty$  and  $b \in M^1$ .

Hence, by letting  $K_{a, \varphi}$  be defined as (3.5) when  $a \in M^\infty$ , it follows that  $a \mapsto K_{a, \varphi}$  on  $M^1$  extends to a continuous map on  $M^\infty$ . Furthermore, since  $\mathcal{S}$  is dense in  $M^\infty$  with respect to the weak\* topology, it follows that this extension is unique. We have therefore proved the theorem for  $p \in \{1, \infty\}$ .

For general  $p \in [1, \infty]$ , the result now follows by interpolation, using Theorem 4.1.2 in [3] and Proposition 1.5.  $\square$

$$(3.1)' \quad s_2 t_1 - s_1 t_2 = 1, \quad |\det(\varphi''_{y, \xi}(x, y, \xi))| \geq d$$

$$(3.2)' \quad \begin{aligned} & \omega_0(s_2 x - t_2 y, -s_1 x + t_1 y, t_1 \xi + s_1 \varphi'_y(x, y, \eta), t_2 \xi + s_2 \varphi'_y(x, y, \eta)) \\ &= \omega(x, \eta, \xi - \varphi'_x(x, y, \eta), -\varphi'_\eta(x, y, \eta)) \end{aligned}$$

and

$$(3.3)' \quad \begin{aligned} & \omega_0(x, y, \xi + s_1 \zeta, \eta + s_2 \zeta) \leq \omega_0(x, y, \xi, \eta) v_1(\zeta), \quad x, y, \xi, \eta, \zeta \in \mathbf{R}^n \\ & \omega(x, \eta, \xi_1 + \xi_2, y_1 + y_2) \leq \omega(x, \eta, \xi_1, y_1) v_2(\xi_2, y_2), \quad x, y_1, y_2, \xi_1, \xi_2, \eta \in \mathbf{R}^n \\ & v(x, y, \zeta, \xi, \eta, z) = v_1(\eta) v_2(\xi, z), \quad x, y, z, \xi, \eta, \zeta \in \mathbf{R}^n. \end{aligned}$$

**Theorem 3.1'.** Assume that  $p \in [1, \infty]$ ,  $s_j, t_j \in \mathbf{R}$  for  $j = 1, 2$ ,  $d > 0$ ,  $v \in \mathcal{P}(\mathbf{R}^{6n})$  is submultiplicative,  $\omega_0, \omega \in \mathcal{P}(\mathbf{R}^{4n})$  and that  $\varphi \in C(\mathbf{R}^{3n})$  are such that  $\varphi$  is real-valued,  $\partial^\alpha \varphi \in M_{(v)}^{\infty, 1}$  for  $|\alpha| = 2$  and (3.1)'–(3.3)' are fulfilled. Then the map

$$a \mapsto K_{a, \varphi}(x, y) \equiv \int a(t_1 x + t_2 y, \xi) e^{i\varphi(t_1 x + t_2 y, s_1 x + s_2 y, \xi)} d\xi,$$

from  $\mathcal{S}(\mathbf{R}^{2n})$  to  $\mathcal{S}'(\mathbf{R}^{2n})$  extends uniquely to a continuous map from  $M_{(\omega)}^p(\mathbf{R}^{2n})$  to  $M_{(\omega_0)}^p(\mathbf{R}^{2n})$ .

*Proof.* By letting

$$x_1 = t_1 x + t_2 y, \quad y_1 = s_1 x + s_2 y$$

as new coordinates, it follows that we may assume that  $t_1 = s_2 = 1$  and  $t_2 = s_1 = 0$ , and then the result agrees with Theorem 3.1. The proof is complete.  $\square$

Assume that  $a \in M^\infty(\mathbf{R}^{2n})$ ,  $t_1, t_2 \in \mathbf{R}$ , and that  $\varphi \in C(\mathbf{R}^{3n})$  is real-valued and satisfies  $\partial^\alpha \varphi \in M_{(v)}^{\infty, 1}$  for  $|\alpha| = 2$  and (3.1) for some  $d > 0$ . Then we let the Fourier integral operator  $\text{Op}_\varphi(a) = \text{Op}_{\varphi, t_1, t_2}(a)$  be the continuous operator from  $\mathcal{S}(\mathbf{R}^n)$  to  $\mathcal{S}'(\mathbf{R}^n)$  with kernel  $K_{a, \varphi}$  in Theorem 3.1. Furthermore, since the case  $t_1 = 1$  and

$t_2 = 0$  is especially important we set  $\text{Op}_{\varphi,0}(a) = \text{Op}_{\varphi,1,0}(a)$ . The following result is now an immediate consequence of Theorem 3.1 and Theorem 4.3 in [35].

Next we consider Fourier operators when  $a \in M_{(\omega)}^{\infty,1}(\mathbf{R}^{2n})$ . In the following it is natural to consider weights  $\omega_1, \omega_2 \in \mathcal{P}(\mathbf{R}^{2n})$  and  $\omega \in \mathcal{P}(\mathbf{R}^{4n})$  which satisfy

$$(3.6) \quad \frac{\omega_2(s_2x - t_2y, t_1\xi + s_1\varphi'_y(x, y, \eta))}{\omega_1(-s_1x + t_1y, -t_2\xi - \varphi'_2(x, y, \eta))} \leq C\omega(x, \eta, \xi - \varphi'_1(x, y, \eta), -\varphi'_3(x, y, \eta)),$$

for some constant  $C$ .

The proof of the following proposition in the case  $p = \infty$  can be found in [?, 19].

**Proposition 3.2.** *Assume that  $p \in [1, \infty]$ ,  $\omega_j \in \mathcal{P}(\mathbf{R}^{2n_j})$ , for  $j = 1, 2$ , and  $\omega \in \mathcal{P}(\mathbf{R}^{2n_1+2n_2})$  fulfill for some positive constant  $C$*

$$\frac{\omega_2(x, \xi)}{\omega_1(y, -\eta)} \leq C\omega(x, y, \xi, \eta).$$

*Assume moreover that  $K \in M_{(\omega)}^p(\mathbf{R}^{n_1+n_2})$  and  $T$  is the linear and continuous map from  $\mathcal{S}(\mathbf{R}^{n_1})$  to  $\mathcal{S}'(\mathbf{R}^{n_2})$  defined by:*

$$(Tf)(x) = (K(x, \cdot), f), \quad f \in \mathcal{S}(\mathbf{R}^{n_1}).$$

*Then  $T$  extends uniquely to a continuous map from  $M_{(\omega_1)}^{p'}(\mathbf{R}^{n_1})$  to  $M_{(\omega_2)}^p(\mathbf{R}^{n_2})$*

*Proof.* By Proposition 1.1 (3) and duality, it suffices to prove that for some constant  $C$  independent of  $f \in \mathcal{S}(\mathbf{R}^{n_1})$  and  $g \in \mathcal{S}(\mathbf{R}^{n_2})$ , it holds:

$$|(K, g \otimes \bar{f})| \leq C\|K\|_{M_{(\omega)}^p}\|g\|_{M_{(1/\omega_2)}^{p'}}\|f\|_{M_{(\omega_1)}^{p'}}.$$

Let us set  $\omega_3(x, \xi) = \omega_1(x, -\xi)$ , then by straightforward calculation and using Remark 1.3 (7) we get

$$\begin{aligned} |(K, g \otimes \bar{f})| &\leq C_1\|K\|_{M_{(\omega)}^p}\|g \otimes \bar{f}\|_{M_{(1/\omega)}^{p'}} \leq C_2\|K\|_{M_{(\omega)}^p}\|g\|_{M_{(1/\omega_2)}^{p'}}\|\bar{f}\|_{M_{(\omega_3)}^{p'}} \\ &\leq C\|K\|_{M_{(\omega)}^p}\|g\|_{M_{(1/\omega_2)}^{p'}}\|f\|_{M_{(\omega_1)}^{p'}} \end{aligned}$$

□

**Proposition 3.3.** *Assume that,  $1 < p < \infty$ ,  $d > 0$ ,  $v \in \mathcal{P}(\mathbf{R}^{6n})$  is submultiplicative,  $\omega_1, \omega_2 \in \mathcal{P}(\mathbf{R}^{2n})$  and  $\omega \in \mathcal{P}(\mathbf{R}^{4n+2m})$  are such that (0.4) is fulfilled for some constant  $C$ , and that  $a \in M_{(\omega)}^{\infty,1}(\mathbf{R}^{2n+m})$ . Also assume that  $\varphi \in C(\mathbf{R}^{3n})$  are such that  $\varphi$  is real-valued,  $\partial^\alpha \varphi \in M_{(v)}^{\infty,1}$  for  $|\alpha| = 2$  and (0.4) are fulfilled. Then  $\text{Op}_{\varphi,0}(a)$  extends to a continuous operator from  $M_{(\omega_1)}^p(\mathbf{R}^n)$  to  $M_{(\omega_2)}^p(\mathbf{R}^n)$ , and*

$$\|\text{Op}_{\varphi,0}(a)f\|_{M_{(\omega_2)}^p} \leq Cd^{-1}\|a\|_{M_{(\omega)}^{\infty,1}}\|f\|_{M_{(\omega_1)}^p} \exp(C\|\varphi''\|_{M_{(v)}^{\infty,1}}).$$

*Proof.* We shall mainly follow the proof of Theorem 3.2?? in [?]. Assume that  $f, g \in \mathcal{S}(\mathbf{R}^n)$ , and that  $0 \leq \chi, \psi \in C_0^\infty(\mathbf{R}^{2n+m})$  and  $0 \leq \chi_0 \in C_0^\infty(\mathbf{R}^n)$  are such that

$$\|\chi_0\|_{L^1} = \|\chi\|_{L^2} = 1$$

and  $\psi = 1$  on  $\text{supp } \chi$ . Also let  $X = (x, y, \zeta) \in \mathbf{R}^{2n+m}$  and  $X_1 = (x_1, y_1, \zeta_1) \in \mathbf{R}^{2n+m}$  as usual. By straight-forward computations we get

$$\begin{aligned} (\text{Op}_\varphi(a)f, g) &= \int a(X)f(y)\overline{g(x)}e^{i\varphi(X)}dX \\ &= \iint a(X + X_1)\chi(X_1)^2f(y + y_1)\chi_0(y_1)\overline{g(x + x_1)\chi_0(x_1)}e^{i\psi(X_1)\varphi(X + X_1)}dXdX_1 \end{aligned}$$

If  $\mathcal{F}_{1,2}a$  denotes the partial Fourier transform of  $a(x, y, \zeta)$  with respect to the  $x$  and  $y$  variables, then Parseval's formula gives

$$\begin{aligned}
& (\text{Op}_\varphi(a)f, g) \\
&= \iiint F(X, \xi, \eta, \zeta_1) \mathcal{F}(f(y + \cdot)\chi_0)(-\eta) \overline{\mathcal{F}(g(x + \cdot)\chi_0)(\xi)} dX d\xi d\eta d\zeta_1 \\
&= \iiint F(X, \xi, \eta, \zeta_1) (V_{\chi_0}f)(y, -\eta) \overline{(V_{\chi_0}g)(x, \xi)} e^{-i(\langle x, \xi \rangle + \langle y, \eta \rangle)} dX d\xi d\eta d\zeta_1, \\
&= \iiint \left( \int F(X, \xi, \eta, \zeta_1) d\zeta_1 \right) (V_{\chi_0}f)(y, -\eta) \overline{(V_{\chi_0}g)(x, \xi)} e^{-i(\langle x, \xi \rangle + \langle y, \eta \rangle)} dX d\xi d\eta,
\end{aligned}$$

where

$$F(X, \xi, \eta, \zeta_1) = \mathcal{F}_{1,2}(e^{i\psi(\cdot, \zeta_1)\varphi(X+(\cdot, \zeta_1))} a(X + (\cdot, \zeta_1)) \chi(\cdot, \zeta_1)^2)(\xi, \eta).$$

In order to further reformulate the action of  $\text{Op}_\varphi(a)$ , we let  $\psi_{1,X}$  and  $\psi_{2,X}$  be the same as in (2.2), and set

$$\begin{aligned}
& \mathcal{H}_{a,\varphi}(X, \xi, \eta) \\
(3.7) \quad &= h_X * (\mathcal{F}(a(\cdot + X)\chi))(\xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X)), \quad \text{where}
\end{aligned}$$

$$h_X = (2\pi)^{-(n+m/2)} (\mathcal{F}(e^{i\psi_{2,X}} \chi))$$

Next let  $\psi_{1,X}$  and  $\psi_{2,X}$  be the same as in (2.2). Then

$$\begin{aligned}
& \int F(X, \xi, \eta, \zeta_1) d\zeta_1 \\
&= \mathcal{F}((e^{i\psi_{2,X}} \chi)(a(\cdot + X)\chi))(\xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X)) \\
&= \mathcal{H}_{a,\varphi}(X, \xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X)),
\end{aligned}$$

where

$$\begin{aligned}
(3.8) \quad & \mathcal{H}_{a,\varphi}(X, \xi, \eta) \\
&= (2\pi)^{-(n+m/2)} (\mathcal{F}(e^{i\psi_{2,X}} \chi)) * (\mathcal{F}(a(\cdot + X)\chi))(X, \xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X)).
\end{aligned}$$

Summing up we have proved that

$$(3.9) \quad (\text{Op}_\varphi(a)f, g) = \iiint \mathcal{H}_{a,\varphi}(X, \xi, \eta) (V_{\chi_0}f)(y, -\eta) \overline{(V_{\chi_0}g)(x, \xi)} e^{-i(\langle x, \xi \rangle + \langle y, \eta \rangle)} dX d\xi d\eta$$

Since

$$|\mathcal{F}(a(\cdot + X)\chi)(\xi, \eta, z)| = |(V_\chi a)(X, \xi, \eta, z)|$$

we therefore get

$$\begin{aligned}
& \left| \int F(X, \xi, \eta, \zeta_1) d\zeta_1 \right| \leq (|\mathcal{F}(e^{i\psi_{2,X}} \chi)| * |V_\chi a(X, \cdot)|)(\xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X)) \\
& \leq \sum_{k=0}^{\infty} \frac{1}{k!} (|\widehat{\chi}| * \Phi_{k,X} * |V_\chi a(X, \cdot)|)(\xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X)),
\end{aligned}$$

where

$$\Phi_{0,X} = \delta_0$$

$$\Phi_{k,X} = |\mathcal{F}(\psi_{2,X})| * \cdots * |\mathcal{F}(\psi_{2,X})|, \quad k \geq 1,$$

with  $k$  factors in the latter convolution.

By Lemma 3.1 it follows that there is a function  $G \in L^1_{(v)}$  such that  $|\mathcal{F}(\psi_{2,X})| \leq G$  and  $\|G\|_{L^1_{(v)}} \leq C\|\varphi''\|_{M^{\infty,1}_{(v)}}$  for some constant  $C > 0$ . Hence if  $\Upsilon_k \equiv G * \dots * G$  with  $k$  factors of  $G$  in the convolution, then it follows that  $|\Phi_{k,X}| \leq \Upsilon_k$  and that

$$\|\Upsilon_k\|_{L^1_{(v)}} \leq C_1^k \|G\|_{L^1_{(v)}}^k \leq C_2^k \|G\|_{M^1_{(v)}}^k \leq C_3^k \|\varphi''\|_{M^{\infty,1}_{(v)}}^k,$$

for some constants  $C_1, \dots, C_3$ . Here the first inequality in the latter estimate follows from the fact that  $v$  is submultiplicative.

By letting

$$J_0(X, \xi, \eta) = (|\widehat{\chi}| * |V_\chi a(X, \cdot)|)(\xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X))$$

$$J_k(X, \xi, \eta) = (|\widehat{\chi}| * \Upsilon_k * |V_\chi a(X, \cdot)|)(\xi - \varphi'_x(X), \eta - \varphi'_y(X), -\varphi'_\zeta(X)), \quad k \geq 1,$$

it follows now that

$$\left| \int F(X, \xi, \eta, \zeta_1) d\zeta_1 \right| \leq C \sum_{k=0}^{\infty} J_k(X, \xi, \eta).$$

Let  $a_1(x, y, \zeta) = a(x, \zeta)$ . For some real-valued function  $\Theta$ , we have

$$\begin{aligned} |(\text{Op}_{\varphi,0}(a)f, g)| &= (2\pi)^{-n} \left| \iiint e^{i\varphi} V_{\chi_3} a_1(X, \xi, \eta, -\varphi'_3(X)) \right. \\ &\quad \left. V_\chi f(y, -\eta - \varphi'_2(X)) \overline{V_\chi g(x, \xi + \varphi'_1(X))} e^{i\Theta(X, \xi, \eta)} d\xi d\eta dX \right| \\ &\leq C \iiint |V_{\chi_2} a(x, \zeta, \xi, -\varphi'_3(X)) \widehat{\chi}(\eta) V_\chi f(y, -\eta - \varphi'_2(X)) \overline{V_\chi g(x, \xi + \varphi'_1(X))}| d\xi d\eta dX \end{aligned}$$

In order to include the conditions for weight functions we set

$$H(\xi, y) = \sup_{x, \zeta} |(V_{\chi_2} a)(x, \zeta, \xi, y) \omega(x, \zeta, \xi, y)|$$

$$F_1(x, \xi) = |(V_\chi f)(x, \xi) \omega_1(x, \xi)|, \quad F_2(x, \xi) = |(V_\chi g)(x, \xi) (\omega_2(x, \xi))^{-1}|,$$

and we note that

$$\|H\| = \|a\|_{M^{\infty,1}_{(\omega)}}, \quad \|F_1\|_{L^p} = \|f\|_{M^p_{(\omega_1)}}, \quad \|F_1\|_{L^{p'}} = \|f\|_{M^{p'}_{(1/\omega_2)}}.$$

By taking  $\xi + \varphi'_1(X)$ ,  $\eta$  and  $X$  as new variables of integration, and using (3.6) we obtain

$$\begin{aligned} |(\text{Op}_{\varphi,0}(a)f, g)| &\leq C \iiint |V_{\chi_2} a(x, \zeta, \xi, -\varphi'_3(X)) \omega(x, \zeta, \xi - \varphi'_1(X), -\varphi'_3(X))| \cdot \\ &\quad |V_\chi f(y, -\eta - \varphi'_2(X)) \omega_1(y, -\varphi'_2(X))| |V_\chi g(x, \xi + \varphi'_1(X)) (\omega_2(x, \xi))^{-1}| |\widehat{\chi}(\eta)| d\xi d\eta dX \\ &\leq C \iiint H(\xi - \varphi'_1(X), -\varphi'_3(X)) F_2(x, \xi) |V_\chi f(y, -\eta - \varphi'_2(X)) \omega_1(y, -\varphi'_2(X))| |\widehat{\chi}(\eta)| d\xi d\eta dX. \end{aligned}$$

Since  $\omega_1$  belongs to  $\mathcal{P}(\mathbf{R}^{2n})$ , it follows that

$$\omega_1(y, -\varphi'_2(X)) \leq \omega_1(y, -\eta - \varphi'_2(X)) v_1(\eta),$$

for some  $v_1 \in \mathcal{P}(\mathbf{R}^n)$ , giving that

$$\begin{aligned} &V_\chi f(y, -\eta - \varphi'_2(X)) \omega_1(y, -\varphi'_2(X)) |\widehat{\chi}(\eta)| \\ &\leq V_\chi f(y, -\eta - \varphi'_2(X)) \omega_1(y, -\eta - \varphi'_2(X)) |\widehat{\chi}(\eta) v_1(\eta)| \leq F_1(y, -\eta - \varphi'_2(X)) h(\eta), \end{aligned}$$

where  $h(\eta) = |\widehat{\chi}(\eta) v_1(\eta)| \in L^1$ .

A combination of these estimates and Hölder's inequality give  
(3.10)

$$|(\text{Op}_{\varphi,0}(a)f, g)| \leq C \iiint H(\xi - \varphi'_1(X), -\varphi'_3(X)) F_1(y, -\eta - \varphi'_2(X)) F_2(x, \xi) h(\eta) d\xi d\eta dX \leq C J_1 \cdot J_2,$$

where

$$J_1^p = \iiint H(\xi - \varphi'_1(X), -\varphi'_3(X)) F_1(y, -\eta - \varphi'_2(X))^p h(\eta) d\xi d\eta dX,$$

$$J_2^{p'} = \iiint H(\xi - \varphi'_1(X), -\varphi'_3(X)) F_2(x, -\xi)^{p'} h(\eta) d\xi d\eta dX$$

We have to estimate  $J_1$  and  $J_2$ . By taking  $x_0 = \varphi'_3(X)$ ,  $\zeta_0 = \varphi'_2(X)$ ,  $y$ ,  $\xi$  and  $\eta$  as new variables of integrations, and using (0.4), it follows that

$$\begin{aligned} J_1 &\leq \left( d^{-1} \int \cdots \int H(\xi - \varphi'_1(X), -x_0) F_1(y, -\eta - \zeta_0)^p h(\eta) dx_0 dy d\xi d\eta d\zeta_0 \right)^{1/p} \\ &= (d^{-1} \|H\|_{L^1} \|h\|_{L^1})^{1/p} \|F_1\|_{L^p}. \end{aligned}$$

Hence

$$(3.11) \quad J_1 \leq (C_h d^{-1} \|a\|_{M_{(\omega)}^{\infty,1}})^{1/p} \|f\|_{M_{(\omega_1)}^p}.$$

If we instead take  $x$ ,  $y_0 = \varphi'_3(X)$ ,  $\xi$ ,  $\eta$  and  $\zeta_0 = \varphi'_1(X)$  as new variables of integrations, it follows by similar arguments that

$$(3.11)' \quad J_2 \leq (C_h d^{-1} \|a\|_{M_{(\omega)}^{\infty,1}})^{1/p'} \|g\|_{M_{(1/\omega_2)}^{p'}}.$$

A combination of (3.10), (3.11) and (3.11)' now give

$$|(\text{Op}_{\varphi,0}(a)f, g)| \leq C_h d^{-1} \|a\|_{M_{(\omega)}^{\infty,1}} \|f\|_{M_{(\omega_1)}^p} \|g\|_{M_{(1/\omega_2)}^{p'}},$$

which proves the assertion.  $\square$

**Theorem 3.4.** *Assume that  $p \in [1, \infty]$ ,  $a \in M^p(\mathbf{R}^{2n})$ ,  $t_1, t_2 \in \mathbf{R}$ , and that  $\varphi \in C(\mathbf{R}^{3n})$  is real-valued and satisfies  $\partial^\alpha \varphi \in M^{\infty,1}$  for  $|\alpha| = 2$  and (3.1) for some  $d > 0$ . Then the definition of  $\text{Op}_{\varphi,t_1,t_2}(a)$  from  $\mathcal{S}(\mathbf{R}^n)$  to  $\mathcal{S}'(\mathbf{R}^n)$  extends uniquely to a continuous map from  $M^{p'}(\mathbf{R}^n)$  to  $M^p(\mathbf{R}^n)$ .*

By combining Theorem ??, Theorem 3.1 and interpolation, we obtain the following result.

**Theorem 3.5.** *Assume that  $p, q \in [1, \infty]$  are such that  $q \leq \min(p, p')$ ,  $a \in M^{p,q}(\mathbf{R}^{2n})$ ,  $t_1, t_2 \in \mathbf{R}$ , and that  $\varphi \in C(\mathbf{R}^{3n})$  is real-valued and satisfies  $\partial^\alpha \varphi \in M^{\infty,1}$  for  $|\alpha| = 2$ . Also assume that (0.4) and (3.1) are fulfilled for some  $d > 0$ . Then the definition of  $\text{Op}_{\varphi,t_1,t_2}(a)$  from  $\mathcal{S}(\mathbf{R}^n)$  to  $\mathcal{S}'(\mathbf{R}^n)$  extends uniquely to a continuous map from  $M^{p'}(\mathbf{R}^n)$  to  $M^p(\mathbf{R}^n)$ . Then the definition of  $\text{Op}_{\varphi,t_1,t_2}(a)$  from  $\mathcal{S}(\mathbf{R}^n)$  to  $\mathcal{S}'(\mathbf{R}^n)$  extends uniquely to a Schatten-von Neumann operator of order  $p$  on  $L^2(\mathbf{R}^n)$ .*

*Proof.* We may assume that  $q = \min(p, p')$ . First assume that  $p \leq 2$ , and let  $b \in \mathcal{S}'(\mathbf{R}^{2n})$  be chosen such that  $b(x, D) = \text{Op}_\varphi(a)$ . Then the operator kernel of  $b$  belongs to  $M^p$ , and since  $M^p$  is invariant under partial Fourier transformations in view of Remark 1.3 (5), the result is a consequence of Proposition 1.7 in [35].

If instead  $p = \infty$ , then it follows from Theorem ?? that  $\text{Op}_\varphi(a)$  is continuous on  $L^2$ , which proves the result in this case as well. The result now follows for general  $p \in [2, \infty]$  by interpolation, using Proposition 1.5 and (1.5). The proof is complete.  $\square$

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